

BASICLU User Guide

Version 2.2

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1 Algorithm

BASICLU implements a right-looking LU factorization with dynamic Markowitz search and columnwise threshold pivoting. After a column modification to the matrix it applies either a permutation or the Forrest-Tomlin update to maintain a factorized form. It uses the method of Gilbert and Peierls to solve triangular systems with a sparse right-hand side. A more detailed explanation of the method is given in [Technical Report ERGO 17-002, <http://www.maths.ed.ac.uk/ERGO/preprints.html>].

2 Installation

2.1 Compiling BASICLU

Compiling BASICLU requires GNUmake and a C compiler that (partly) supports the ANSI C99 standard.

To compile the package type `make` in the BASICLU directory. This will create a static and a shared library inside `lib/`. It will also compile a standalone program `maxvolume` in `example/`. You can call the latter with the matrices in `example/data/`.

Compiler and linker flags can be changed in `config.mk` or can be given to `make` on the command line. See the documentation in `config.mk`.

2.2 The integer type

BASICLU integer variables are of type `lu_int`, which is `typedef`'ed in `basiclu.h`. `lu_int` must be a signed integer type. The default is `int64_t`. It can be changed before compiling the package. Note:

- The BASICLU routines do not check for integer overflow. It is in your responsibility to choose a sufficiently large integer type for your problems.
- It is required that all integer values arising in the computation can be stored in `double` variables and converted back to `lu_int` without altering their value.

3 Low level C interface

The low level C interface consists of routines which do not allocate memory. Memory must be provided by the user and reallocated on request. To use the low level C interface, user code must include `basiclu.h`. Defined constants start with `BASICLU_` and function names start with `basiclu_`.

Memory must be provided in form of four `lu_int` arrays and four `double` arrays:

`istore`, `xstore` are arrays whose size depends only on the matrix dimension (see `basiclu_initialize` for their required length). `xstore` is used to input parameters to the routines and to return information to the user. The indices of `xstore` which the user may access have defined names, e. g. `xstore[BASICLU_STATUS]` holds the status code. `istore` need not be accessed by the user.

`Li`, `Lx`, `Ui`, `Ux`, `Wi`, `Wx` are arrays whose required size is not known in advance. Their size must be given by the user as parameters (see below) and BASICLU will request reallocation if the size is insufficient. These arrays need not be accessed by the user.

3.1 basiclu_initialize

```
lu_int basiclu_initialize
(
    lu_int m,
    lu_int istore[],
    double xstore[]
);
```

Purpose:

Initialize istore, xstore to a BASICLU instance. Set parameters to defaults and reset counters. The initialization fixes the dimension of matrices which can be processed by this instance.

This routine must be called once before passing istore, xstore to any other basiclu_ routine.

Return:

BASICLU_OK

m, istore, xstore were valid arguments. Only in this case are istore, xstore initialized.

BASICLU_ERROR_argument_missing

istore or xstore is NULL.

BASICLU_ERROR_invalid_argument

m is less than or equal to zero.

Arguments:

lu_int m

The dimension of matrices which can be processed. $m > 0$.

lu_int istore[]

double xstore[]

Fixed size arrays. These must be allocated by the user as follows:

length of istore: $\text{BASICLU_SIZE_ISTORE_1} + \text{BASICLU_SIZE_ISTORE_M} * m$

length of xstore: $\text{BASICLU_SIZE_XSTORE_1} + \text{BASICLU_SIZE_XSTORE_M} * m$

Info:

After initialization, the following entries of xstore are maintained throughout by all basiclu_ routines:

xstore[BASICLU_DIM] Matrix dimension (constant).

xstore[BASICLU_NUPDATE] Number of updates since last factorization. This is the sum of Forrest-Tomlin updates and permutation updates.

xstore[BASICLU_NFORREST] Number of Forrest-Tomlin updates since last factorization. The upper limit on Forrest-Tomlin updates before refactorization is m, but that is far too much for performance reasons and numerical stability.

xstore[BASICLU_NFACTORIZE] Number of factorizations since initialization.

xstore[BASICLU_NUPDATE_TOTAL] Number of updates since initialization.

xstore[BASICLU_NFORREST_TOTAL] Number of Forrest-Tomlin updates since initialization.

xstore[BASICLU_NSYPERM_TOTAL] Number of symmetric permutation updates since initialization. A permutation update is "symmetric" if the row and column permutation can be updated symmetrically.

xstore[BASICLU_LNZ] Number of nonzeros in L excluding diagonal elements (not changed by updates).

xstore[BASICLU_UNZ] Number of nonzeros in U excluding diagonal elements (changed by updates).

xstore[BASICLU_RNZ] Number of nonzeros in update ETA vectors excluding diagonal elements (zero after factorization, increased by Forrest-Tomlin updates).

xstore[BASICLU_MIN_PIVOT]
xstore[BASICLU_MAX_PIVOT] After factorization these are the smallest and largest pivot element. xstore[BASICLU_MIN_PIVOT] is replaced when a smaller pivot occurs in an update. xstore[BASICLU_MAX_PIVOT] is replaced when a larger pivot occurs in an update.

xstore[BASICLU_UPDATE_COST] Deterministic measure of solve/update cost compared to cost of last factorization. This value is zero after factorization and monotonically increases with solves/updates. When xstore[BASICLU_UPDATE_COST] > 1.0, then a refactorization is good for performance.

xstore[BASICLU_TIME_FACTORIZE] Wall clock time for last factorization.

xstore[BASICLU_TIME_SOLVE] Wall clock time for all calls to basiclu_solve_sparse and basiclu_solve_for_update since last factorization.

xstore[BASICLU_TIME_UPDATE] Wall clock time for all calls to basiclu_update since last factorization.

xstore[BASICLU_TIME_FACTORIZE_TOTAL]
xstore[BASICLU_TIME_SOLVE_TOTAL]
xstore[BASICLU_TIME_UPDATE_TOTAL] Analogous to above, but summing up all calls since initialization.

xstore[BASICLU_LFLOPS]
xstore[BASICLU_UFLOPS]
xstore[BASICLU_RFLOPS] Number of flops for operations with L, U and update ETA vectors in calls to basiclu_solve_sparse and basiclu_solve_for_update since last factorization.

3.2 basiclu_factorize

```
lu_int basiclu_factorize
(
    lu_int istore[],
    double xstore[],
    lu_int Li[],
    double Lx[],
    lu_int Ui[],
    double Ux[],
    lu_int Wi[],
    double Wx[],
    const lu_int Bbegin[],
    const lu_int Bend[],
    const lu_int Bi[],
    const double Bx[],
    lu_int cContinue
);
```

Purpose:

Factorize the matrix B into its LU factors. Choose pivot elements by a Markowitz criterion subject to columnwise threshold pivoting (the pivot may not be smaller than a factor of the largest entry in its column).

Return:

BASICLU_ERROR_invalid_store if istore, xstore do not hold a BASICLU instance. In this case xstore[BASICLU_STATUS] is not set.

Otherwise return the status code. See xstore[BASICLU_STATUS] below.

Arguments:

```
lu_int istore[]
double xstore[]
```

BASICLU instance. The instance determines the dimension of matrix B (stored in xstore[BASICLU_DIM]).

```
lu_int Li[]
double Lx[]
lu_int Ui[]
double Ux[]
lu_int Wi[]
double Wx[]
```

Arrays used for workspace during the factorization and to store the final factors. They must be allocated by the user and their length must be provided as parameters:

```
xstore[BASICLU_MEMORYL]: length of Li and Lx
xstore[BASICLU_MEMORYU]: length of Ui and Ux
xstore[BASICLU_MEMORYW]: length of Wi and Wx
```

When the allocated length is insufficient to complete the factorization, basiclu_factorize() returns to the caller for reallocation (see xstore[BASICLU_STATUS] below). A successful factorization requires at least nnz(B) length for each of the arrays.

```
const lu_int Bbegin[]
const lu_int Bend[]
const lu_int Bi[]
const double Bx[]
```

Matrix B in packed column form. Bi and Bx are arrays of row indices

and nonzero values. Column j of matrix B contains elements

$B_i[Bbegin[j] .. Bend[j]-1], B_x[Bbegin[j] .. Bend[j]-1]$.

The columns must not contain duplicate row indices. The arrays $Bbegin$ and $Bend$ may overlap, so that it is valid to pass $Bp, Bp+1$ for a matrix stored in compressed column form (Bp, Bi, Bx).

lu_int cContinue

zero to start a new factorization; nonzero to continue a factorization after reallocation.

Parameters:

xstore[BASICLU_DROP_TOLERANCE]

Nonzeros whose magnitude is less than or equal to the drop tolerance can be removed after each pivot step. They are guaranteed removed at the end of the factorization. Default: $1e-20$

xstore[BASICLU_ABS_PIVOT_TOLERANCE]

A pivot element must be nonzero and in absolute value must be greater than or equal to $xstore[BASICLU_ABS_PIVOT_TOLERANCE]$. Default: $1e-14$

xstore[BASICLU_REL_PIVOT_TOLERANCE]

A pivot element must be (in absolute value) greater than or equal to $xstore[BASICLU_REL_PIVOT_TOLERANCE]$ times the largest entry in its column. A value greater than or equal to 1.0 is treated as 1.0 and enforces partial pivoting. Default: 0.1

xstore[BASICLU_BIAS_NONZEROS]

When this value is greater than or equal to zero, the pivot choice attempts to keep L sparse, putting entries into U when possible. When this value is less than zero, the pivot choice attempts to keep U sparse, putting entries into L when possible. Default: 1

xstore[BASICLU_MAXN_SEARCH_PIVOT]

The Markowitz search is terminated after searching $xstore[BASICLU_MAXN_SEARCH_PIVOT]$ rows or columns if a numerically stable pivot element has been found. Default: 3

xstore[BASICLU_SEARCH_ROWS]

If $xstore[BASICLU_SEARCH_ROWS]$ is zero, then the Markowitz search only scans columns. If nonzero, then both columns and rows are searched in increasing order of number of entries. Default: 1

xstore[BASICLU_PAD]

xstore[BASICLU_STRETCH]

When a row or column cannot be updated by the pivot operation in place, it is appended to the end of the workspace. For a row or column with nz elements, $xstore[BASICLU_PAD] + nz * xstore[BASICLU_STRETCH]$ elements extra space are added for later fill-in. Default: $xstore[BASICLU_PAD] = 4, xstore[BASICLU_STRETCH] = 0.3$

xstore[BASICLU_REMOVE_COLUMNS]

This parameter is present for compatibility to previous versions but has no effect. If during factorization the maximum entry of a column of the

active submatrix becomes zero or less than `xstore[BASICLU_ABS_PIVOT_TOLERANCE]`, then that column is immediately removed without choosing a pivot.

Info:

`xstore[BASICLU_STATUS]`: status code.

`BASICLU_OK`

The factorization has successfully completed.

`BASICLU_WARNING_singular_matrix`

The factorization did `xstore[BASICLU_RANK] < xstore[BASICLU_DIM]` pivot steps. The remaining elements in the active submatrix are zero or less than `xstore[BASICLU_ABS_PIVOT_TOLERANCE]`. The factors have been augmented by unit columns to form a square matrix. See `basiclu_get_factors()` on how to get the indices of linearly dependent columns.

`BASICLU_ERROR_argument_missing`

One or more of the pointer/array arguments are NULL.

`BASICLU_ERROR_invalid_call`

`cContinue` is nonzero, but the factorization was not started before.

`BASICLU_ERROR_invalid_argument`

The matrix is invalid (a column has a negative number of entries, a row index is out of range, or a column has duplicate entries).

`BASICLU_REALLOCATE`

Factorization requires more memory in `Li,Lx` and/or `Ui,Ux` and/or `Wi,Wx`. The number of additional elements in each of the array pairs required for the next pivot operation is given by:

```
xstore[BASICLU_ADD_MEMORYL] >= 0
xstore[BASICLU_ADD_MEMORYU] >= 0
xstore[BASICLU_ADD_MEMORYW] >= 0
```

The user must reallocate the arrays for which additional memory is required. It is recommended to reallocate for the requested number of additional elements plus some extra space (e.g. 0.5 times the current array length). The new array lengths must be provided in

```
xstore[BASICLU_MEMORYL]: length of Li and Lx
xstore[BASICLU_MEMORYU]: length of Ui and Ux
xstore[BASICLU_MEMORYW]: length of Wi and Wx
```

`basiclu_factorize()` can be called again with `cContinue` not equal to zero to continue the factorization.

`xstore[BASICLU_MATRIX_NZ]` number of nonzeros in B

`xstore[BASICLU_MATRIX_ONENORM]`

`xstore[BASICLU_MATRIX_INFNUM]` 1-norm and inf-norm of the input matrix after replacing dependent columns by unit columns.

`xstore[BASICLU_RANK]` number of pivot steps performed

`xstore[BASICLU_BUMP_SIZE]` dimension of matrix after removing singletons
`xstore[BASICLU_BUMP_NZ]` # nonzeros in matrix after removing singletons
`xstore[BASICLU_NSEARCH_PIVOT]` total # columns/rows searched for pivots
`xstore[BASICLU_NEXPAND]` # columns/rows which had to be appended to the end
of the workspace for the rank-1 update
`xstore[BASICLU_NGARBAGE]` # garbage collections
`xstore[BASICLU_FACTOR_FLOPS]` # floating point operations performed,
counting multiply-add as one flop
`xstore[BASICLU_TIME_SINGLETONS]` wall clock time for removing the initial
triangular factors
`xstore[BASICLU_TIME_SEARCH_PIVOT]` wall clock time for Markowitz search
`xstore[BASICLU_TIME_ELIM_PIVOT]` wall clock time for pivot elimination
`xstore[BASICLU_RESIDUAL_TEST]`

An estimate for numerical stability of the factorization.
`xstore[BASICLU_RESIDUAL_TEST]` is the maximum of the scaled residuals

$$\|b-Bx\| / (\|b\| + \|B\|* \|x\|)$$

and

$$\|c-B'y\| / (\|c\| + \|B'\|* \|y\|),$$

where $x=B\backslash b$ and $y=B'\backslash c$ are computed from the LU factors, b and c
have components ± 1 that are chosen to make x respectively y large,
and $\|\cdot\|$ is the 1-norm. Here B is the input matrix after replacing
dependent columns by unit columns.

If `xstore[BASICLU_RESIDUAL_TEST] > 1e-12`, say, the factorization is
numerically unstable. (This is independent of the condition number
of B .) In this case tightening the relative pivot tolerance and
refactorizing is appropriate.

`xstore[BASICLU_NORM_L]`
`xstore[BASICLU_NORM_U]` 1-norm of L and U .

`xstore[BASICLU_NORMEST_LINV]`
`xstore[BASICLU_NORMEST_UINV]` Estimated 1-norm of L^{-1} and U^{-1} ,
computed by the LINPACK algorithm.

`xstore[BASICLU_CONDEST_L]`
`xstore[BASICLU_CONDEST_U]` Estimated 1-norm condition number of L and U .

3.3 basiclu_get_factors

```
lu_int basiclu_get_factors
(
    lu_int istore[],
    double xstore[],
    lu_int Li[],
    double Lx[],
    lu_int Ui[],
    double Ux[],
    lu_int Wi[],
    double Wx[],
    lu_int rowperm[],
    lu_int colperm[],
    lu_int Lcolptr[],
    lu_int Lrowidx[],
    double Lvalue[],
    lu_int Ucolptr[],
    lu_int Urowidx[],
    double Uvalue[]
);
```

Purpose:

Extract the row and column permutation and the LU factors. This routine can be used only after `basiclu_factorize()` has completed and before a call to `basiclu_update()`. At that point the factorized form of matrix B is

$$B[\text{rowperm}, \text{colperm}] = L*U,$$

where L is unit lower triangular and U is upper triangular. If the factorization was singular ($\text{rank} < m$), then columns `colperm[rank..m-1]` of B have been replaced by unit columns with entry 1 in position `rowperm[rank..m-1]`.

`basiclu_get_factors()` is intended when the user needs direct access to the matrix factors. It is not required to solve linear systems with the factors (see `basiclu_solve_dense()` and `basiclu_solve_sparse()` instead).

Return:

`BASICLU_ERROR_invalid_store` if `istore`, `xstore` do not hold a `BASICLU` instance. In this case `xstore[BASICLU_STATUS]` is not set.

Otherwise return the status code. See `xstore[BASICLU_STATUS]` below.

Arguments:

```
lu_int istore[]
double xstore[]
lu_int Li[]
double Lx[]
lu_int Ui[]
double Ux[]
lu_int Wi[]
double Wx[]
```

The `BASICLU` instance after `basiclu_factorize()` has completed.

```
lu_int rowperm[m]
```

Returns the row permutation. If the row permutation is not required, then `NULL` can be passed (this is not an error).

```
lu_int colperm[m]
```

Returns the column permutation. If the column permutation is not required, then NULL can be passed (this is not an error).

```
lu_int Lcolptr[m+1]
lu_int Lrowidx[m+Lnz]
double Lvalue[m+Lnz], where Lnz = xstore[BASICLU_LNZ]
```

If all three arguments are not NULL, then they are filled with L in compressed column form. The indices in each column are sorted with the unit diagonal element at the front.

If any of the three arguments is NULL, then L is not returned (this is not an error).

```
lu_int Ucolptr[m+1]
lu_int Urowidx[m+Unz]
double Uvalue[m+Unz], where Unz = xstore[BASICLU_UNZ]
```

If all three arguments are not NULL, then they are filled with U in compressed column form. The indices in each column are sorted with the diagonal element at the end.

If any of the three arguments is NULL, then U is not returned (this is not an error).

Info:

xstore[BASICLU_STATUS]: status code.

BASICLU_OK

The requested quantities have been returned successfully.

BASICLU_ERROR_argument_missing

One or more of the mandatory pointer/array arguments are NULL.

BASICLU_ERROR_invalid_call

The BASICLU instance does not hold a fresh factorization (either basiclu_factorize() has not completed or basiclu_update() has been called in the meanwhile).

3.4 basiclu_solve_dense

```
lu_int basiclu_solve_dense
(
    lu_int istore[],
    double xstore[],
    lu_int Li[],
    double Lx[],
    lu_int Ui[],
    double Ux[],
    lu_int Wi[],
    double Wx[],
    const double rhs[],
    double lhs[],
    char trans
);
```

Purpose:

Given the factorization computed by `basiclu_factorize()` or `basiclu_update()` and the dense right-hand side, `rhs`, solve a linear system for the solution `lhs`.

Return:

`BASICLU_ERROR_invalid_store` if `istore`, `xstore` do not hold a `BASICLU` instance. In this case `xstore[BASICLU_STATUS]` is not set.

Otherwise return the status code. See `xstore[BASICLU_STATUS]` below.

Arguments:

```
lu_int istore[]
double xstore[]
lu_int Li[]
double Lx[]
lu_int Ui[]
double Ux[]
lu_int Wi[]
double Wx[]
```

Factorization computed by `basiclu_factorize()` or `basiclu_update()`.

```
const double rhs[m]
```

The right-hand side vector.

```
double lhs[m]
```

Uninitialized on entry. On return `lhs` holds the solution to the linear system.

`lhs` and `rhs` are allowed to overlap. To overwrite `rhs` with the solution pass pointers to the same array.

```
char trans
```

Defines which system to solve. 't' or 'T' for the transposed system, any other character for the forward system.

Info:

`xstore[BASICLU_STATUS]`: status code.

`BASICLU_OK`

The linear system has been successfully solved.

BASICLU_ERROR_argument_missing

One or more of the pointer/array arguments are NULL.

BASICLU_ERROR_invalid_call

The factorization is invalid.

3.5 basiclu_solve_sparse

```
lu_int basiclu_solve_sparse
(
    lu_int istore[],
    double xstore[],
    lu_int Li[],
    double Lx[],
    lu_int Ui[],
    double Ux[],
    lu_int Wi[],
    double Wx [],
    lu_int nzrhs,
    const lu_int irhs[],
    const double xrhs[],
    lu_int *p_nzlhs,
    lu_int ilhs[],
    double lhs[],
    char trans
);
```

Purpose:

Given the factorization computed by `basiclu_factorize()` or `basiclu_update()` and the sparse right-hand side, `rhs`, solve a linear system for the solution `lhs`.

Return:

`BASICLU_ERROR_invalid_store` if `istore`, `xstore` do not hold a `BASICLU` instance. In this case `xstore[BASICLU_STATUS]` is not set.

Otherwise return the status code. See `xstore[BASICLU_STATUS]` below.

Arguments:

```
lu_int istore[]
double xstore[]
lu_int Li[]
double Lx[]
lu_int Ui[]
double Ux[]
lu_int Wi[]
double Wx[]
```

Factorization computed by `basiclu_factorize()` or `basiclu_update()`.

```
lu_int nzrhs
const lu_int irhs[nzrhs]
const double xrhs[nzrhs]
```

The right-hand side vector in compressed format. `irhs[0..nzrhs-1]` are the indices of nonzeros and `xrhs[0..nzrhs-1]` the corresponding values. `irhs` must not contain duplicates.

```
lu_int *p_nzlhs
lu_int ilhs[m]
lu_int lhs[m]
```

`*p_nzlhs` is uninitialized on entry. On return `*p_nzlhs` holds the number of nonzeros in the solution.
The contents of `ilhs` is uninitialized on entry. On return `ilhs[0..*p_nzlhs-1]` holds the indices of nonzeros in the solution.
The contents `lhs` must be initialized to zero on entry. On return the solution is scattered into `lhs`.

char trans

Defines which system to solve. 't' or 'T' for the transposed system, any other character for the forward system.

Parameters:

xstore[BASICLU_SPARSE_THRESHOLD]

Defines which method is used for solving a triangular system. A triangular solve can be done either by the two phase method of Gilbert and Peierls ("sparse solve") or by a sequential pass through the vector ("sequential solve").

Solving $B*x=b$ requires two triangular solves. The first triangular solve is done sparse. The second triangular solve is done sparse if its right-hand side has not more than $m * \text{xstore[BASICLU_SPARSE_THRESHOLD]}$ nonzeros. Otherwise the sequential solve is used.

Default: 0.05

xstore[BASICLU_DROP_TOLERANCE]

Nonzeros which magnitude is less than or equal to the drop tolerance are removed after each triangular solve. Default: $1e-20$

Info:

xstore[BASICLU_STATUS]: status code.

BASICLU_OK

The linear system has been successfully solved.

BASICLU_ERROR_argument_missing

One or more of the pointer/array arguments are NULL.

BASICLU_ERROR_invalid_call

The factorization is invalid.

BASICLU_ERROR_invalid_argument

The right-hand side is invalid ($nzrhs < 0$ or $nzrhs > m$ or one or more indices out of range).

3.6 basiclu_solve_for_update

```
lu_int basiclu_solve_for_update
(
    lu_int istore[],
    double xstore[],
    lu_int Li[],
    double Lx[],
    lu_int Ui[],
    double Ux[],
    lu_int Wi[],
    double Wx[],
    lu_int nzrhs,
    const lu_int irhs[],
    const double xrhs[],
    lu_int *p_nzlhs,
    lu_int ilhs[],
    double lhs[],
    char trans
);
```

Purpose:

Given the factorization computed by `basiclu_factorize()` or `basiclu_update()`, solve a linear system in preparation to update the factorization.

When the forward system is solved, then the right-hand side is the column to be inserted into the factorized matrix. When the transposed system is solved, then the right-hand side is a unit vector with entry 1 in position of the column to be replaced in the factorized matrix.

For BASICLU to prepare the update, it is sufficient to compute only a partial solution. If the left-hand side is not requested by the user (see below), then only one triangular solve is done. If the left-hand side is requested, then a second triangular solve is required.

Return:

`BASICLU_ERROR_invalid_store` if `istore`, `xstore` do not hold a BASICLU instance. In this case `xstore[BASICLU_STATUS]` is not set.

Otherwise return the status code. See `xstore[BASICLU_STATUS]` below.

Arguments:

```
lu_int istore[]
double xstore[]
lu_int Li[]
double Lx[]
lu_int Ui[]
double Ux[]
lu_int Wi[]
double Wx[]
```

Factorization computed by `basiclu_factorize()` or `basiclu_update()`.

```
lu_int nzrhs
const lu_int irhs[nzrhs]
const double xrhs[nzrhs]
```

The right-hand side vector in compressed format.

When the forward system is solved, `irhs[0..nzrhs-1]` are the indices of nonzeros and `xrhs[0..nzrhs-1]` the corresponding values. `irhs` must not contain duplicates.

When the transposed system is solved, the right-hand side is a unit vector with entry 1 in position `irhs[0]`. `nzrhs`, `xrhs` and elements of `irhs` other than `irhs[0]` are not accessed. `xrhs` can be NULL.

```
lu_int *p_nzlhs
lu_int ilhs[m]
lu_int lhs[m]
```

If any of `p_nzlhs`, `ilhs` or `lhs` is NULL, then the solution to the linear system is not requested. In this case only the update is prepared.

Otherwise:

`*p_nzlhs` is uninitialized on entry. On return `*p_nzlhs` holds the number of nonzeros in the solution.
The contents of `ilhs` is uninitialized on entry. On return `ilhs[0..*p_nzlhs-1]` holds the indices of nonzeros in the solution.
The contents of `lhs` must be initialized to zero on entry. On return the solution is scattered into `lhs`.

```
char trans
```

Defines which system to solve. 't' or 'T' for the transposed system, any other character for the forward system.

Parameters:

```
xstore[BASICLU_MEMORYL]: length of Li and Lx
xstore[BASICLU_MEMORYU]: length of Ui and Ux
xstore[BASICLU_MEMORYW]: length of Wi and Wx
```

```
xstore[BASICLU_SPARSE_THRESHOLD]
```

Defines which method is used for solving a triangular system. A triangular solve can be done either by the two phase method of Gilbert and Peierls ("sparse solve") or by a sequential pass through the vector ("sequential solve").

When the solution to the linear system is requested, then two triangular systems are solved. The first triangular solve is done sparse. The second triangular solve is done sparse if its right-hand side has not more than $m * \text{xstore[BASICLU_SPARSE_THRESHOLD]}$ nonzeros. Otherwise the sequential solve is used.

When the solution to the linear system is not requested, then this parameter has no effect.

Default: 0.05

```
xstore[BASICLU_DROP_TOLERANCE]
```

Nonzeros which magnitude is less than or equal to the drop tolerance are removed after each triangular solve. Default: $1e-20$

Info:

```
xstore[BASICLU_STATUS]: status code.
```

```
BASICLU_OK
```

The updated has been successfully prepared and, if requested, the solution to the linear system has been computed.

```
BASICLU_ERROR_argument_missing
```


One or more of the mandatory pointer/array arguments are NULL.

BASICLU_ERROR_invalid_call

The factorization is invalid.

BASICLU_ERROR_maximum_updates

There have already been m Forrest-Tomlin updates, see `xstore[BASICLU_NFORREST]`. The factorization cannot be updated any more and must be recomputed by `basiclu_factorize()`. The solution to the linear system has not been computed.

BASICLU_ERROR_invalid_argument

The right-hand side is invalid (forward system: $nzrhs < 0$ or $nzrhs > m$ or one or more indices out of range; backward system: `irhs[0]` out of range).

BASICLU_REALLOCATE

The solve was aborted because of insufficient memory in L_i, L_x or U_i, U_x to store data for `basiclu_update()`. The number of additional elements required is given by

```
xstore[BASICLU_ADD_MEMORYL] >= 0
xstore[BASICLU_ADD_MEMORYU] >= 0
```

The user must reallocate the arrays for which additional memory is required. It is recommended to reallocate for the requested number of additional elements plus some extra space for further updates (e.g. 0.5 times the current array length). The new array lengths must be provided in

```
xstore[BASICLU_MEMORYL]: length of  $L_i$  and  $L_x$ 
xstore[BASICLU_MEMORYU]: length of  $U_i$  and  $U_x$ 
```

`basiclu_solve_for_update()` will start from scratch in the next call.

3.7 basiclu_update

```
lu_int basiclu_update
(
    lu_int istore[],
    double xstore[],
    lu_int Li[],
    double Lx[],
    lu_int Ui[],
    double Ux[],
    lu_int Wi[],
    double Wx[],
    double xtbl
);
```

Purpose:

Update the factorization to replace one column of the factorized matrix. A call to `basiclu_update()` must be preceded by calls to `basiclu_solve_for_update()` to provide the column to be inserted and the index of the column to be replaced.

The column to be inserted is defined as the right-hand side in the last call to `basiclu_solve_for_update()` in which the forward system was solved.

The index of the column to be replaced is defined by the unit vector in the last call to `basiclu_solve_for_update()` in which the transposed system was solved.

Return:

`BASICLU_ERROR_invalid_store` if `istore`, `xstore` do not hold a `BASICLU` instance. In this case `xstore[BASICLU_STATUS]` is not set.

Otherwise return the status code. See `xstore[BASICLU_STATUS]` below.

Arguments:

```
lu_int istore[]
double xstore[]
lu_int Li[]
double Lx[]
lu_int Ui[]
double Ux[]
lu_int Wi[]
double Wx[]
```

Factorization computed by `basiclu_factorize()` or `basiclu_update()`.

```
double xtbl
```

This is an optional argument to monitor numerical stability. `xtbl` can be either of

- (a) element `j0` of the solution to the forward system computed by `basiclu_solve_for_update()`, where `j0` is the column to be replaced;
- (b) the dot product of the incoming column and the solution to the transposed system computed by `basiclu_solve_for_update()`.

In either case `xstore[BASICLU_PIVOT_ERROR]` (see below) has a defined value. If monitoring stability is not desired, `xtbl` can be any value.

Parameters:

`xstore[BASICLU_MEMORYL]`: length of `Li` and `Lx`

xstore[BASICLU_MEMORYU]: length of U_i and U_x
xstore[BASICLU_MEMORYW]: length of W_i and W_x

xstore[BASICLU_DROP_TOLERANCE]

Nonzeros which magnitude is less than or equal to the drop tolerance are removed from the row eta matrix. Default: $1e-20$

Info:

xstore[BASICLU_STATUS]: status code.

BASICLU_OK

The update has successfully completed.

BASICLU_ERROR_argument_missing

One or more of the pointer/array arguments are NULL.

BASICLU_ERROR_invalid_call

The factorization is invalid or the update was not prepared by two calls to `basiclu_solve_for_update()`.

BASICLU_REALLOCATE

Insufficient memory in W_i, W_x . The number of additional elements required is given by

$xstore[BASICLU_ADD_MEMORYW] > 0$

The user must reallocate W_i, W_x . It is recommended to reallocate for the requested number of additional elements plus some extra space for further updates (e.g. 0.5 times the current array length). The new array length must be provided in

$xstore[BASICLU_MEMORYW]$: length of W_i and W_x

`basiclu_update` will start from scratch in the next call.

BASICLU_ERROR_singular_update

The updated factorization would be (numerically) singular. No update has been computed and the old factorization is still valid.

xstore[BASICLU_PIVOT_ERROR]

When `xtbl` was given (see above), then $xstore[BASICLU_PIVOT_ERROR]$ is a measure for numerical stability. It is the difference between two computations of the new pivot element relative to the new pivot element. A value larger than $1e-10$ indicates numerical instability and suggests refactorization (and possibly tightening the pivot tolerance).

xstore[BASICLU_MAX_ETA]

The maximum entry (in absolute value) in the eta vectors from the Forrest-Tomlin update. A large value, say $> 1e6$, indicates that pivoting on diagonal element was unstable and refactorization might be necessary.

4 High level C interface

The high level C interface consists of wrapper functions around the low level interface which do memory allocation. They maintain the arrays used by the low level interface inside a `struct basiclu_object`. To use the high level C interface, user code must include `basiclu.h`. Defined constants start with `BASICLU_` and function names start with `basiclu_obj_`.

4.1 basiclu_object

```
struct basiclu_object
{
    lu_int *istore;
    double *xstore;
    lu_int *Li, *Ui, *Wi;
    double *Lx, *Ux, *Wx;
    double *lhs;
    lu_int *ilhs;
    lu_int nzlhs;
    double realloc_factor;
};
```

A variable of type struct basiclu_object must be defined in user code. Its members are set and maintained by basiclu_obj_* routines. User code should only access the following members:

xstore (read/write)

set parameters and get info values

lhs, ilhs, nzlhs (read only)

holds solution after solve_sparse() and solve_for_update()

realloc_factor (read/write)

Arrays are reallocated for max(realloc_factor, 1.0) times the required size. Default: 1.5

4.2 basiclu_obj_initialize

```
lu_int basiclu_obj_initialize
(
    struct basiclu_object *obj,
    lu_int m
);
```

Purpose:

Initialize a BASICLU object. When *m* is positive, then **obj* is initialized to process matrices of dimension *m*. When *m* is zero, then **obj* is initialized to a "null" object, which cannot be used for factorization, but can be passed to `basiclu_obj_free()`.

This routine must be called once before passing *obj* to any other `basiclu_obj_` routine. When *obj* is initialized to a null object, then the routine can be called again to reinitialize *obj*.

Return:

BASICLU_OK

**obj* successfully initialized.

BASICLU_ERROR_argument_missing

obj is NULL.

BASICLU_ERROR_invalid_argument

m is negative.

BASICLU_ERROR_out_of_memory

insufficient memory to initialize object.

Arguments:

struct basiclu_object *obj

Pointer to the object to be initialized.

lu_int m

The dimension of matrices which can be processed, or 0.

4.3 basiclu_obj_get_dim

```
lu_int basiclu_obj_get_dim  
(  
    struct basiclu_object *obj  
);
```

Purpose:

Returns the dimension for which the object was initialized or 0 when obj is NULL.

4.4 basiclu_obj_factorize

```
lu_int basiclu_obj_factorize
(
    struct basiclu_object *obj,
    const lu_int *Bbegin,
    const lu_int *Bend,
    const lu_int *Bi,
    const double *Bx
);
```

Purpose:

Call `basiclu_factorize()` on a BASICLU object.

Return:

`BASICLU_ERROR_invalid_object`

`obj` is NULL or initialized to a null object.

`BASICLU_ERROR_out_of_memory`

reallocation failed because of insufficient memory.

Other return codes are passed through from `basiclu_factorize()`.

Arguments:

`struct basiclu_object *obj`

Pointer to an initialized BASICLU object.

The other arguments are passed through to `basiclu_factorize()`.

4.5 basiclu_obj_get_factors

```
lu_int basiclu_obj_get_factors
(
    struct basiclu_object *obj,
    lu_int rowperm[],
    lu_int colperm[],
    lu_int Lcolptr[],
    lu_int Lrowidx[],
    double Lvalue[],
    lu_int Ucolptr[],
    lu_int Urowidx[],
    double Uvalue[]
);
```

Purpose:

Call `basiclu_get_factors()` on a BASICLU object.

Return:

`BASICLU_ERROR_invalid_object`

`obj` is `NULL` or initialized to a null object.

Other return codes are passed through from `basiclu_get_factors()`.

Arguments:

`struct basiclu_object *obj`

Pointer to an initialized BASICLU object.

The other arguments are passed through to `basiclu_get_factors()`.

4.6 basiclu_obj_solve_dense

```
lu_int basiclu_obj_solve_dense
(
    struct basiclu_object *obj,
    const double rhs[],
    double lhs[],
    char trans
);
```

Purpose:

Call `basiclu_solve_dense()` on a BASICLU object.

Return:

`BASICLU_ERROR_invalid_object`

`obj` is `NULL` or initialized to a null object.

Other return codes are passed through from `basiclu_solve_dense()`.

Arguments:

`struct basiclu_object *obj`

Pointer to an initialized BASICLU object.

The other arguments are passed through to `basiclu_solve_dense()`.

4.7 basiclu_obj_solve_sparse

```
lu_int basiclu_obj_solve_sparse
(
    struct basiclu_object *obj,
    lu_int nzrhs,
    const lu_int irhs[],
    const double xrhs[],
    char trans
);
```

Purpose:

Call `basiclu_solve_sparse()` on a BASICLU object. On success, the solution is provided in `obj->lhs` and the nonzero pattern is stored in `obj->ilhs[0..obj->nzlhs-1]`.

Return:

`BASICLU_ERROR_invalid_object`

`obj` is `NULL` or initialized to a null object.

Other return codes are passed through from `basiclu_solve_sparse()`.

Arguments:

`struct basiclu_object *obj`

Pointer to an initialized BASICLU object.

The other arguments are passed through to `basiclu_solve_sparse()`.

4.8 basiclu_obj_solve_for_update

```
lu_int basiclu_obj_solve_for_update
(
    struct basiclu_object *obj,
    lu_int nzrhs,
    const lu_int irhs[],
    const double xrhs[],
    char trans,
    lu_int want_solution
);
```

Purpose:

Call `basiclu_solve_for_update()` on a BASICLU object. On success, if the solution was requested, it is provided in `obj->lhs` and the nonzero pattern is stored in `obj->ilhs[0..obj->nzlhs-1]`.

Return:

`BASICLU_ERROR_invalid_object`

`obj` is NULL or initialized to a null object.

`BASICLU_ERROR_out_of_memory`

reallocation failed because of insufficient memory.

Other return codes are passed through from `basiclu_solve_for_update()`.

Arguments:

`struct basiclu_object *obj`

Pointer to an initialized BASICLU object.

`lu_int want_solution`

Nonzero to compute the solution to the linear system,
zero to only prepare the update.

The other arguments are passed through to `basiclu_solve_for_update()`.

4.9 basiclu_obj_update

```
lu_int basiclu_obj_update
(
    struct basiclu_object *obj,
    double xtbl
);
```

Purpose:

Call `basiclu_update()` on a BASICLU object.

Return:

`BASICLU_ERROR_invalid_object`

`obj` is `NULL` or initialized to a null object.

`BASICLU_ERROR_out_of_memory`

reallocation failed because of insufficient memory.

Other return codes are passed through from `basiclu_update()`.

Arguments:

`struct basiclu_object *obj`

Pointer to an initialized BASICLU object.

The other arguments are passed through to `basiclu_update()`.

4.10 basiclu_obj_free

```
void basiclu_obj_free  
(  
    struct basiclu_object *obj  
);
```

Purpose:

Free memory allocated from a BASICLU object. The object must have been initialized before by `basiclu_obj_initialize()`. Subsequent calls to `basiclu_obj_free()` will do nothing.

Arguments:

```
struct basiclu_object *obj
```

Pointer to the object which memory is to be freed. When `obj` is `NULL`, then the routine does nothing.

4.11 basiclu_obj_maxvolume

```
lu_int basiclu_obj_maxvolume
(
    struct basiclu_object *obj,
    lu_int ncol,
    const lu_int Ap[],
    const lu_int Ai[],
    const double Ax[],
    lu_int basis[],
    lu_int isbasic[],
    double volumetol,
    lu_int *p_nupdate
);
```

Purpose:

Make one pass over the columns of a rectangular ($ncol \geq nrow$) matrix and pivot each nonbasic column into the basis when it increases the volume (i.e. the absolute value of the determinant) of the basis matrix. This is one main loop of the "maximum volume" algorithm described in [1,2].

[1] C. T. Pan, "On the existence and computation of rank-revealing LU factorizations". *Linear Algebra Appl.*, 316(1-3), pp. 199-222, 2000

[2] S. A. Goreinov, I. V. Oseledets, D. V. Savostyanov, E. E. Tyrtyshnikov, N. L. Zamarashkin, "How to find a good submatrix". In "Matrix methods: theory, algorithms and applications", pp. 247-256. World Sci. Publ., Hackensack, NJ, 2010.

Return:

BASICLU_ERROR_invalid_argument when volumetol is less than 1.0.
BASICLU_ERROR_out_of_memory when memory allocation in this function failed.

The return code from a basiclu_obj_* function called when not BASICLU_OK.
(Note that BASICLU_WARNING_singular_matrix means that the algorithm failed.)

BASICLU_OK otherwise.

Arguments:

```
struct basiclu_object *obj
```

Pointer to an initialized BASICLU object. The dimension of the object specifies the number of rows of the matrix.

```
lu_int ncol
const lu_int Ap[ncol+1]
const lu_int Ai[]
const double Ax[]
```

Matrix A in compressed sparse column format. Column j contains elements

$Ai[Ap[j] .. Ap[j+1]-1]$, $Ax[Ap[j] .. Ap[j+1]-1]$.

The columns must not contain duplicate row indices. The row indices per column need not be sorted.

```
lu_int basis[nrow]
```

On entry holds the column indices of A that form the initial basis. On return holds the updated basis. A basis defines a square nonsingular submatrix of A. If the initial basis is (numerically) singular, then the initial LU factorization will fail and BASICLU_WARNING_singular_matrix is returned.

lu_int isbasic[ncol]

This array must be consistent with basis[] on entry, and is consistent on return. isbasic[j] must be nonzero iff column j appears in the basis.

double volumetol

A column is pivoted into the basis when it increases the absolute value of the determinant of the basis matrix by more than a factor volumetol. This parameter must be ≥ 1.0 . In practice typical values are 2.0, 10.0 or even 100.0. The closer the tolerances to 1.0, the more basis changes will usually be necessary to find a maximum volume basis for this tolerance (using repeated calls to basiclu_obj_maxvolume(), see below).

lu_int *p_nupdate

On return *p_nupdate holds the number of basis updates performed. When this is zero and BASICLU_OK is returned, then the volume of the initial basis is locally (within one basis change) maximum up to a factor volumetol. To find such a basis, basiclu_obj_maxvolume() must be called repeatedly starting from an arbitrary basis until *p_nupdate is zero. This will happen eventually because each basis update strictly increases the volume of the basis matrix. Hence a basis cannot repeat.

p_nupdate can be NULL, in which case it is not accessed. This is not an error condition. The number of updates performed can be obtained as the increment to obj->xstore[BASICLU_NUPDATE_TOTAL] caused by the call to basiclu_obj_maxvolume().

5 Julia interface

BASICLU can be used from the Julia programming language. The easiest way is to install the artifact `basiclu_jll`, which provides a precompiled library. Alternatively the code can be compiled from source as described in Section 2. In this case ensure that `int64_t` is used as integer type (this is the default). To use the locally compiled library, environment variable `JULIA_BASICLU_LIBRARY_PATH` must point to the `lib/` directory.

The following is an example for a Julia program using BASICLU. See also the documentation of the module functions and `Julia/test.jl`.

```
include("BASICLU/Julia/basiclu.jl")
m = 1000
obj = basiclu.basiclu_object(m)
B = sprand(m,m,5e-3) + I           # get a sparse matrix
basiclu.factorize(obj, B)
rhs = randn(m)                    # get a right-hand side
lhs = basiclu.solve(obj, rhs, 'N')
res = norm(B*lhs-rhs,Inf)         # compute residual
col = sparsevec([1], [1.0], m)    # vector to be inserted into B
lhs = basiclu.solve_for_update(obj, col, getsol=true)
vmax, j = findmax(abs.(lhs))
piv = lhs[j]
basiclu.solve_for_update(obj, j)  # prepare to replace column j of B
piverr = basiclu.update(obj, piv)
lhs = basiclu.solve(obj, rhs, 'N')
B[:,j] = col
res = norm(B*lhs-rhs,Inf)
```