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Complexity of minimum biclique cover and minimum biclique decomposition for bipartite domino-free graphs

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Abstract

A biclique cover (resp. biclique decomposition) of a bipartite graph B is a family of complete bipartite subgraphs of B whose edges cover (resp. partition) the edges of B. The minimum cardinality of a biclique cover (resp. biclique decomposition) is denoted by *s-dim*(B) (resp. *s-part*(B)). The decision problems associated with the computation of *s-dim* and *s-part* are NP-complete for general bipartite graphs; the decision problem associated to *s-dim* is NP-complete for bipartite chordal graphs, and polynomial for bipartite distance-hereditary graphs, for bipartite convex graphs and for bipartite C4-free graphs. We show here that for bipartite domino-free graphs (a strict generalization of bipartite distance-hereditary graphs and bipartite C4-free graphs), *s-dim* and *s-part* are equal and can be computed in $O(n \times m)$ time. Moreover, we propose a $O(n \times m)$ time algorithm to check the domino-free property and to build the Galois lattice of such graphs.

Keywords: Bipartite domino-free graphs; Complete bipartite graphs (or bicliques); Minimum biclique cover; Minimum biclique decomposition; Galois lattice

1. Introduction

The problem of covering the edges of a bipartite graph by complete bipartite subgraphs (or *bicliques* for short) arises in many areas: automata and language theories [7], graphs [5], partial orders [8] and has applications for graph compression [12], in artificial intelligence [18] and in biology [16].

A *biclique cover* (resp. *biclique decomposition*) of a graph G is a family of complete bipartite subgraphs of G whose edges cover (resp. partition) the edges of G. The minimum cardinality of a biclique cover (also called bipartite dimension of G or $d(G)$ in [5], $\kappa_e(G)$ in [15], $\beta(G)$ in [12] and set-dimension of G in [18] for G bipartite) is denoted by *s-dim*(G); in the following, *s-part*(G) denotes the minimum cardinality of

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a biclique decomposition of G. Throughout this paper, we only consider biclique cover and biclique decomposition of bipartite graphs.

In fact, it has been shown that computing *s-dim*(B) for a bipartite graph B is polynomially equivalent to

- computing the ambiguous rank of a boolean matrix [7].
- computing the size of a minimum boolean basis of a 0–1 matrix M (that is the minimum number of blocks of M – submatrix which are constantly 1 – such that each 1 entry in M belongs to at least one of these blocks) [12].
- computing the 2-dimension of a lattice L (that is the smallest k such that L has an embedding in $\{0,1\}^k$) [5, 8].
- computing the smallest edge coloring of B such that two non-incident edges not belonging both to an induced cycle of length 4 (or for short $C4$) have different colors [5, 15].
- computing the minimal number of specificities required in order to explain a given set of human leukocyte antigen reactions [16].

The decision problem corresponding to *s-dim* has been proved NP-complete for general bipartite graphs [17] and even for chordal bipartite graphs [15].

To our knowledge, the largest classes for which the *s-dim* problem is polynomial are:

1. Bipartite C4-free graphs, which are bipartite graphs without induced subgraph isomorphic to the $C4$ [15].
2. Bipartite distance-hereditary graphs [15]. Several characterizations exist for this class [4]; we take here the following: a bipartite graph B is distance-hereditary if and only if every cycle of length greater or equal than 6 has at least two chords.
3. Bipartite convex graphs [6, 12, 13]. We take the following characterization: B is a bipartite convex graph if and only if its adjency matrix has the consecutive 1's property [4], i.e. is an interval matrix [12]. The proof for this class can be deduced from the trivial statement that the minimum biclique cover problem on bipartite convex graphs is exactly the minimum boolean basis problem on interval matrix, and, from [13] which shows that: "the boolean basis problem on interval matrix is exactly the interval basis problem and is well solved by [6]".

The decision problem associated with *s-part* has also been proved NP-complete for general bipartite graphs [9], but does not seem to have been widely studied otherwise.

In the following, a *domino* is the 6-cycle with exactly one chord shown in Fig. 1. Bipartite domino-free graphs are graphs without induced subgraph isomorphic to a domino. By definition, neither bipartite $C4$-free graphs nor bipartite distance-hereditary graphs have any induced domino. Inversely, a bipartite domino-free graph may have induced $C4$, or cycles of length greater or equal than 6 without chords: so the class of bipartite domino-free graphs is a strict generalization of the two former classes.

The main result of this paper is that for any bipartite domino-free graph B, *s-dim* and *s-part* are equal and can be computed in $O(n \times m)$ time, where n and m are respectively the number of vertices and the number of edges of B. To our knowledge,

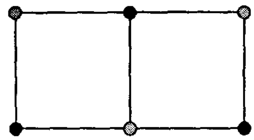

Fig. 1. The domino.

it is the first non trivial polynomial class allowing chordless cycles. Fig. 7 shows the hierarchy of several classes of bipartite graphs (extracted from [4]) and presents for each of them the computational complexity of the decision problems associated with *s-dim* and *s-part*.

Incidently, as the jump number of a bipartite distance-hereditary graph is polynomially related to its *s-dim* parameter (cf. [14]), our algorithm improves the $O(m^2)$ time algorithm proposed in [14] to compute the jump number parameter for these graphs.

We have shown in [2] that for bipartite domino-free graphs the number of maximal bicliques is at most m; the same property holds for bipartite chordal graphs [10], so this is not a sufficient condition to ensure the polynomiality of the *s-dim* problem. In the case of domino-free bipartite graphs, their Galois lattices have structural properties which allow to compute *s-dim* in polynomial time; besides, we can build this lattice in $O(n \times m)$ time.

The remaining sections are organized as follows. Section 2 gives definitions and notations. Section 3 gives some fundamental properties of bipartite domino-free graphs. Section 4 establishes that we can restrict the problem to a subclass: the bipartite domino-free *simplified* graphs. Section 5 then shows how to compute *s-dim* by using the Galois lattice associated with these graphs. Finally, Sections 6 and 7 give $O(n \times m)$ time algorithms to check the domino-free property and to build the associated Galois lattice.

2. Definitions and notation

Throughout this paper, a bipartite graph B is a finite, simple and undirected graph defined by (X_B, Y_B, E_B). X_B and Y_B partition the vertices of B into two independent sets. E_B denotes the set of edges of B. The number of vertices and the number of edges of B are respectively denoted by n and m.

For any vertex x of B we denote its neighborhood by $N(x) = \{y \mid \{x, y\} \in E_B\}$, that is the set of all vertices adjacent to x; its degree is denoted by $d(x) = |N(x)|$. By $I(x)$ we denote the set of edges incident with x, $I(x) = \{\{x, y\} \in E_B \mid y \in N(x)\}$.

The bipartite subgraph of B induced by $X \subseteq X_B$ and $Y \subseteq Y_B$ is denoted by $B(X, Y)$.

A *biclique* K of B is a pair (X, Y) such that $X \subseteq X_B$, $Y \subseteq Y_B$, X and Y are nonempty and $B(X, Y)$ is a complete bipartite subgraph of B. The set of edges of K is $X \times Y$ and is denoted by E_K.

$\mathscr{K}(B)=\{K_i=(X_i,Y_i)\}$ (resp. $\mathscr{K}_M(B)\subseteq\mathscr{K}(B)$) is the set of all bicliques (resp. maximal bicliques) of B.

For any $K_1,K_2\in\mathscr{K}(B)$, K_2-K_1 is the subgraph of B defined by $(X(E_{K_2}\backslash E_{K_1}), Y(E_{K_2}\backslash E_{K_1}), E_{K_2}\backslash E_{K_1})$.

A *star* is an n-vertex graph with $n-1$ vertices of degree 1 and 1 vertex of degree $n-1$, the center of the star.

$\mathscr{S}(B)\subseteq\mathscr{K}(B)$ (resp. $\mathscr{S}_M(B)\subseteq\mathscr{K}_M(B)$) is the set of all stars (resp. maximal stars) of B; $\overline{\mathscr{S}}(B)=\mathscr{K}(B)\backslash\mathscr{S}(B)$ (resp. $\overline{\mathscr{S}_M}(B)=\mathscr{K}_M(B)\backslash\mathscr{S}_M(B)$) is the set of all non-star bicliques (resp. maximal non-star bicliques) of B.

A *partially ordered set* will be denoted by $P=(V,<_p)$, where V is the ground set of elements and $<_p$ is the *order relation* i.e., an antisymmetric, antireflexive and transitive relation whose element $(x,y)\in <_p$ are written as $x<_p y$ $(x,y\in V)$ with the usual interpretation. When we talk about the reflexive closure of $<_p$, we use the notation $\leqslant_p$. Two elements $x,y\in V$ are *comparable* in P if $x\leqslant_p y$ or $y\leqslant_p x$; otherwise they are said to be *incomparable*. A *total order* is a partial order without incomparable pairs. The *transitive reduction* of a partially ordered set P is the directed acyclic graph with vertex set V and arc set E with $(x,y)\in E$ if and only if $x<_p y$ and there is no $z\in V$ such that $x<_p z$ and $z<_p y$; it is sometimes called the Hasse diagram of P.

3. Some properties of bipartite domino-free graphs

3.1. s-dim and s-part are equal for bipartite domino-free graphs

Property 3.1. *Let B be a bipartite graph and $K_1=(X_1,Y_1),K_2=(X_2,Y_2)\in\mathscr{K}_M(B)$ two maximal bicliques of B such that $K_1\neq K_2$. Then $X_1\subset X_2\Leftrightarrow Y_2\subset Y_1$.*

Theorem 3.1. *Let B be a bipartite graph. Then B is domino-free if and only if $\forall K_1,K_2\in\mathscr{K}_M(B)$ such that $K_1\neq K_2$ and $E_{K_1}\cap E_{K_2}\neq\emptyset$, one of these statements is true:* (i) *$X_1\subset X_2$ and $Y_2\subset Y_1$*; (ii) *$X_2\subset X_1$ and $Y_1\subset Y_2$*

Proof. ($\Rightarrow$) Let K_1 and K_2 be two maximal bicliques sharing a common edge $\{x,y\}$ and such that (i) and (ii) are false. From Property 3.1 we can deduce (a) $X_1\backslash X_2\neq\emptyset$, (b) $X_2\backslash X_1\neq\emptyset$, (c) $Y_1\backslash Y_2\neq\emptyset$, (d) $Y_2\backslash Y_1\neq\emptyset$.

- pick x_1 in $X_1\backslash X_2$ (a) and y_2 in $Y_2\backslash Y_1$ (d) such that $\{x_1,y_2\}\notin E_B$ (if $Y_2\backslash Y_1\subseteq N(x_1)$ then, as $Y_1\cap Y_2\subseteq N(x_1)$, $Y_2\subseteq N(x_1)$ and K_2 is not a maximal biclique);
  pick x_2 in $X_2\backslash X_1$ (b) and y_1 in $Y_1\backslash Y_2$ (c) such that $\{x_2,y_1\}\notin E_B$.

Then, $\{x,y,x_1,y_1\}$ and $\{x,y,x_2,y_2\}$ induce two C4 of B that share the edge $\{x,y\}$ and, as $\{x_1,y_2\}$ and $\{x_2,y_1\}\notin E_B$, $\{x,y,x_1,y_1,x_2,y_2\}$ is the domino of Fig. 2.

($\Leftarrow$) If B has a domino induced by $\{x,y,x_1,y_1,x_2,y_2\}$ with chord $\{x,y\}$ (cf. Fig. 2), there is $K_1\in\mathscr{K}_M(B)$ such that K_1 contains the $C4$ $\{x,y,x_1,y_1\}$ and $K_2\in\mathscr{K}_M(B)$ containing the $C4$ $\{x,y,x_2,y_2\}$. Since $\{x_1,y_2\}\notin E_B$, $x_1\in X_1\backslash X_2$, so (i) is false. Similarly, we obtain (ii) false. $\square$

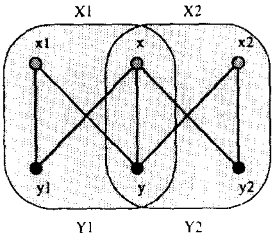

Fig. 2. Proof of Theorem 3.1.

Definition 3.1. Let B be a bipartite graph and $K_1, K_2 \in \mathscr{K}(B)$ such that $K_1 \neq K_2$. We say that K_1 *cut properly* K_2 if and only if $K_2 - K_1$ is a biclique of B or $E_{K_2} \subset E_{K_1}$.

Lemma 3.1. *Let B be a bipartite domino-free graph and $K_2 \in \mathscr{K}(B)$. Then K_2 is properly cut by any $K_1 \in \mathscr{K}_M(B)$ different from K_2.*

Proof. The nontrivial case is for $E_{K_2} \not\subset E_{K_1}$ and K_2 is not a star (the property to be a star is an hereditary property with respect to edge deletion).

Then, let $K_3 \in \mathscr{K}_M(B)$ such that $E_{K_2} \subseteq E_{K_3}$. If $X_3 \subset X_1$ then $X_2 \subset X_1$ and $K_2 - K_1$ is the biclique $(X_2, Y_2 \backslash Y_1)$ of B. If $Y_3 \subset Y_1$ then $Y_2 \subset Y_1$ and $K_2 - K_1$ is the biclique $(X_2 \backslash X_1, Y_2)$ of B. By Theorem 3.1, there is no other case. □

Theorem 3.2. *Let B be a bipartite domino-free graph. Then s-dim(B) = s-part(B).*

Proof. Any biclique decomposition of B is a biclique cover of B. For any minimum biclique cover $C = \{K_1, K_2, \ldots, K_k\} \subseteq \mathscr{K}_M(B)$, $\{K_i - K_{i+1} - K_{i+2} - \cdots - K_k, 1 \leqslant i \leqslant k\}$ is a set of k bicliques of B (Lemma 3.1) that forms a decomposition of B. □

3.2. The Galois lattice of bipartite domino-free graphs

Let $\top = (\emptyset, Y_B)$, $\bot = (X_B, \emptyset)$, and $\mathscr{K}_M^*(B) = \mathscr{K}_M(B) \cup \{\top, \bot\}$. Let $<$ be the order of $\mathscr{K}_M^*(B)$ defined by $\forall K_1, K_2 \in \mathscr{K}_M^*(B), K_2 < K_1 \Leftrightarrow X_1 \subset X_2$ and $Y_2 \subset Y_1$.

The set $\mathscr{K}_M^*(B)$ ordered by $<$ has a structure of lattice. It is known as the *Galois lattice* (or concepts lattice) of B and we denote it by $Gal(B) = (\mathscr{K}_M^*(B), <)$.

Definition 3.2. Let B be a bipartite graph. For every $e \in E_B$, we denote $\mathscr{K}_M(B, e)$ the set of all maximal bicliques of B containing e, $\mathscr{K}_M(B, e) = \{K_i \in \mathscr{K}_M(B) \mid e \in E_{K_i}\}$

The following theorem is only a restatement of Theorem 3.1 using the Galois lattice framework.

Theorem 3.3. *A bipartite graph B is domino-free if and only if for all $e \in E_B$ the restriction of $<$ to $\mathscr{K}_M(B, e)$ is a total order.*

This theorem directly provides an upper bound on the number of maximal bicliques of a bipartite domino-free graph B: $|\mathscr{K}_M(B)| \leqslant n \times m$. It is known that the Galois lattice of a bipartite graph can be computed in $O(|X_B|^2 \times (|X_B| + |Y_B|) \times |\mathscr{K}_M(B)|)$ [3]; so it can be computed in polynomial time for bipartite domino-free graphs.

Corollary 3.1. *Let B be a bipartite domino-free graph, $x \in X_B$ and $K_1, K_2, K_3 \in \mathscr{K}_M(B)$ such that $x \in X_1 \cap X_2 \cap X_3$. If $K_1 < K_2$ and $K_1 < K_3$ then K_2 and K_3 are comparable.*

Proof. From definition of $<$ we can deduce that $Y_1 \subseteq Y_2 \cap Y_3$. Then, for any $y \in Y_2 \cap Y_3$, $\{x, y\} \in E_{K_2} \cap E_{K_3}$ and by Theorem 3.3, K_2 and K_3 are comparable. □

It follows from the previous lemma that the subgraph of the transitive reduction of the Galois lattice of a bipartite domino-free graph B, induced by the set of bicliques covering a given vertex of B, is a tree.

4. Simplifying a bipartite domino-free graph

Let us consider the hypergraph $\mathscr{H} = (\mathscr{K}_M(B), \{\mathscr{K}_M(B,e),\ e \in E_B\})$. Finding a minimum cover of B amounts to find a minimum transversal of the hyperedges of $\mathscr{H}$. It is then natural to consider only the subsets $\mathscr{K}_M(B,e)$ which are minimal with respect to inclusion. The simplification defined here allows us to consider only such subsets.

Definition 4.1. Let B be a bipartite graph. $\leqslant$ is the preorder on $X_B \cup Y_B$ defined by

$$\forall x, x' \in X_B \cup Y_B \text{ such that } N(x) \neq \emptyset, \quad x \leqslant x' \Leftrightarrow N(x) \subseteq N(x')$$

In the following, for $x \in X_B \cup Y_B$, $Succ(x) = \{x' \neq x \text{ s.t. } x \leqslant x'\}$. We can note that the vertices x having a non-empty neighborhood which are maximal with respect to $\leqslant$ (i.e. $Succ(x) = \emptyset$) are exactly the centers of the stars which are maximal bicliques.

This relation induces a relation on E_B, defined by

$$\forall e, e' \in E_B,\ e \leqslant e' \Leftrightarrow \left[\begin{array}{l} e = \{x, y\} \text{ and } e' = \{x, y'\} \text{ and } y \leqslant y' \\ \text{or} \\ e = \{x, y\} \text{ and } e' = \{x', y\} \text{ and } x \leqslant x' \end{array}\right.$$

Property 4.1. *Let B be a bipartite graph and $e \in E_B$.*

$$e \in Min(E_B, \leqslant) \Leftrightarrow \mathscr{K}_M(B,e) \in Min(\{\mathscr{K}_M(B,e') / e' \in E_B\}, \subseteq)$$

Proof. In the following, we take $e = \{x, y\}$.

($\Leftarrow$) Assume that $\exists e' = \{x, y'\} \in E_B$ such that $y \neq y'$ and $e' \leqslant e$. Let $K_i = (X_i, Y_i) \in \mathscr{K}_M(B,e')$. As $N(y') \subseteq N(y)$ and $X_i \subseteq N(y')$, we have $X_i \subseteq N(y)$. By maximality of K_i, then $y \in Y_i$. Therefore $K_i \in \mathscr{K}_M(B,e)$

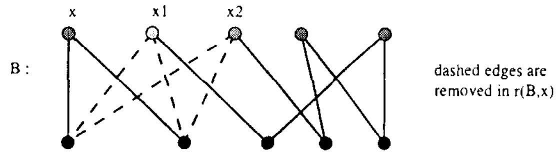

Fig. 3. Reduction operation.

($\Rightarrow$) We assume now that $\exists e' = \{x', y'\} \in E_B$ such that $\mathscr{K}_M(B,e') \subseteq \mathscr{K}_M(B,e)$. Let K_i be the maximal biclique of $\mathscr{K}_M(B,e')$ containing the star of center y' ($X_i = N(y')$). As $K_i \in \mathscr{K}_M(B,e)$, $y \in Y_i$ and then $y' \leqslant y$. The edge $e'' = \{x, y'\}$ is such that $e'' \leqslant e$. □

Definition 4.2 (*Reduction operation*). Let B be a bipartite graph and a vertex x of $X_B \cup Y_B$ such that $N(x) \neq \emptyset$ and $Succ(x) \neq \emptyset$. The graph $r(B,x)$ is the partial graph of B defined by $(X_B, Y_B, E_{r(B,x)})$ with $E_{r(B,x)} = E_B \backslash \{\{x', z\} \mid x' \in Succ(x) \text{ and } z \in N(x)\}$.

Note: Each edge e removed by the reduction operation is a non-minimal edge of B with respect to $\leqslant$ and then, by Property 4.1, for each of those edges, $\mathscr{K}_M(B,e)$ is not minimal with respect to $\subseteq$.

Now, let us establish with the two next properties that as well the parameter *s-dim* as the domino-free property are invariant under the reduction operation for domino-free bipartite graphs.

Property 4.2. *For a bipartite domino-free graph B, s-dim(B) = s-dim(r(B, x)).*

Proof.

Lemma 4.1. *Let B be a bipartite domino-free graph,* $K_i \in \mathscr{K}_M(B)$, *and* $x' \in Succ(x)$ *such that* $x' \in X_i$, $x \notin X_i$, *and* $N(x) \cap Y_i \neq \emptyset$. *Then* $\forall x'' \in X_i$, $x'' \in Succ(x)$.

Proof of Lemma 4.1. We have $Y_i \backslash N(x) \neq \emptyset$ (otherwise, $Y_i \subseteq N(x)$ and the maximality of K_i contradicts $x \notin X_i$). There are two cases:
- if $N(x) \subseteq Y_i$, then $\forall x'' \in X_i, N(x) \subseteq N(x'')$
- if $N(x) \backslash Y_i \neq \emptyset$. Let y be any vertex in $N(x) \backslash Y_i$; we pick $y_1 \in N(x) \cap Y_i$ and $y_2 \in Y_i \backslash N(x)$; then y, y_1, y_2 are all distincts. The set $\{x', x'', y_1, y_2\}$ induces a $C4$. The set $\{x', x, y_1, y\}$ induces a $C4$ too. As $\{x, y_2\} \notin E_B$ (by the choice of y_2), and as B is domino-free, the edge $\{x'', y\} \in E_B$. Then $N(x) \subseteq N(x'')$.

So, in all cases, we have $x'' \in Succ(x)$. □

Proof of Property 4.2. Without loss of generality, we take $x \in X_B$.

We show first that $s\text{-}dim(B) \leqslant s\text{-}dim(r(B,x))$.

Let $\{K'_1, \ldots, K'_i, \ldots, K'_{s\text{-}dim(r(B,x))}\}$ be a minimum cover of $r(B,x)$. We define the set $\{K_1, \ldots K_{s\text{-}dim(r(B,x))}\}$ by

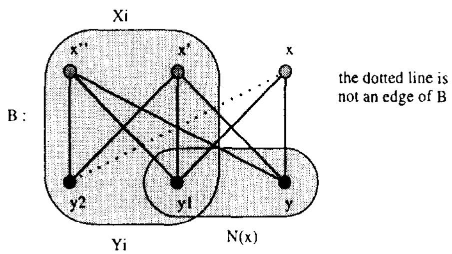

Fig. 4. Proof of Lemma 4.1.

(a) $K_i = K'_i$ if x is not a vertex of K'_i.
(b) $K_i = B(X'_i \cup Succ(x), Y'_i)$ otherwise.

Trivially, any K_i is a biclique of B (for case (b), this follows from definition of $Succ(x)$). Moreover, the set $\{K_1, \dots, K_{s\text{-}dim(r(B,x))}\}$ is a biclique cover of B: the edges of $r(B,x)$ are covered as $\forall i$, $K'_i \subseteq K_i$, and every $\{x', y\} \in E_B \backslash E_{r(B,x)}$ is covered by K_i the biclique obtained from K'_i, with $\{x, y\} \in E_{K'_i}$. So we have: $s\text{-}dim(B) \leqslant s\text{-}dim(r(B,x))$.

Inversely, let $\{K_1, \dots, K_{s\text{-}dim(B)}\}$ be a minimum biclique cover of B where all bicliques belong to $\mathscr{K}_M(B)$. We shall construct $\{K'_1, \dots, K'_{s\text{-}dim(B)}\}$ a biclique cover of $r(B,x)$ as following:
(a) $K'_i = B(X_i \backslash Succ(x), Y_i)$ if $x \in X_i$,
(b) $K'_i = B(X_i, Y_i \backslash N(x))$ if $x \notin X_i$ and $X_i \cap Succ(x) \neq \emptyset$,
(c) $K'_i = K_i$ otherwise.

Clearly, every K'_i is a biclique of B. Moreover, $E_{K'_i} = E_{K_i} \cap E_{r(B,x)}$: in case (a), this follows from the definition of $E_{r(B,x)}$, whereas in case (b), this follows from Lemma 4.1; in case (c), $E_{K_i} \subseteq E_{r(B,x)}$. Then every K'_i is a biclique of $r(B,x)$.

As $\bigcup_{i \in 1..s\text{-}dim(r(B,x))} E_{K'_i} = (\bigcup_{i \in 1..s\text{-}dim(r(B,x))} E_{K_i}) \cap E_{r(B,x)} = E_B \cap E_{r(B,x)} = E_{r(B,x)}$, the set $\{K'_1, \dots K'_{s\text{-}dim(B)}\}$ is a biclique cover of $r(B,x)$, therefore $s\text{-}dim(r(B,x)) \leqslant s\text{-}dim(B)$. □

Property 4.3. *Let B be a bipartite domino-free graph. Then every reduction of B is domino-free too.*

Proof. Let us assume that in the reduction of B there is a domino $\{x, y, x_1, y_1, x_2, y_2\}$ where $\{x, y\}$ is the chord. As B is domino-free, at least one of the edges $\{x_2, y_1\}$ or $\{x_1, y_2\}$ is an edge of B. We suppose that $\{x_1, y_2\}$ belongs to E_B and does not belong to the reduction of B. Without loss of generality, we can suppose that the reduction of B is $r(B, y_3)$ with y_3 such that $N(y_3) \subseteq N(y_2)$ and $\{x_1, y_3\} \in E_B$. Moreover, we have $N(y_3) \nsubseteq N(y_1)$ (otherwise $\{x_1, y_1\}$ could not belong to $r(B, y_3)$), and for the same reason, we have $N(y_3) \nsubseteq N(y)$. Then $\exists x_3 \in N(y_3)$ s.t. $\{x_3, y\} \notin E_B$. As $N(y_3) \subseteq N(y_2)$, $\{x_3, y_2\} \in E_B$. The sets $\{x_1, y_2, x_2, y\}$ and $\{x_1, y_2, x_3, y_3\}$ induce two $C4$ sharing the edge $\{x_1, y_2\}$. As there is no edge $\{x_3, y\}$, and as B is domino-free, the edge $\{x_2, y_3\}$ belongs to E_B. But then, the edge $\{x_2, y_2\}$ cannot belong to $r(B, y_3)$ which contradicts the hypothesis. □

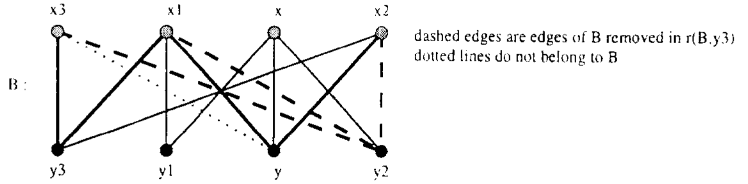

Fig. 5. Proof of Property 4.

Definition 4.3. A bipartite graph is *simplified* when no reduction operation is possible, that is, $\forall x, x' \in X_B \cup Y_B, N_B(x) \not\subseteq N_B(x')$ and $N_B(x') \not\subseteq N_B(x)$. We denote by B^s a graph obtained by applying repeatedly reduction operations until no reduction is possible.

Note: The proof of Property 4.2 gives the way to compute $C \subseteq \mathscr{K}_M(B)$, a minimum biclique cover of a domino-free bipartite graph B, from a given minimum biclique cover $C' \subseteq \mathscr{K}_M(B^s)$ of B^s. This can be done by visiting in reverse order a list of all $(u, Succ(u))$, $u \in X_B \cup Y_B$, used to reduce B: just update C' in extending each bicliques of C' containing u to $Succ(u)$.

Property 4.4. *Let B be a bipartite graph; then if $N_B(y_1) \backslash N_B(y_2) \neq \emptyset$, $\forall x \in X_B$, $N_{r(B,x)}(y_1) \backslash N_{r(B,x)}(y_2) \neq \emptyset$.*

Proof. Let $x_1 \in N_B(y_1) \backslash N_B(y_2)$. If $x_1 \in N_{r(B,x)}(y_1)$, the property is true. If it is not the case, this means $x_1 \in Succ(x)$ and $y_1 \in N_B(x)$. As $\{x_1, y_2\} \notin E_B$, we have $\{x, y_2\} \notin E_B$. Then $x \in N_{r(B,x)}(y_1) \backslash N_{r(B,x)}(y_2)$. □

In other words, reducing B with reduction operations using vertices in X_B does not create new reduction operations for vertices in Y_B (and reciprocally). So, to obtain B^s, we can proceed by performing first all reductions using only vertices of one side of B, and then all reductions using only vertices of the other side.

Algorithm 1: SIMPLIFICATION
Input: B a bipartite graph
Output: B^s

begin
$B' \leftarrow$ ONE SIDE SIMPLIFICATION(B, X_B)
$B^s \leftarrow$ ONE SIDE SIMPLIFICATION(B', Y_B)
end

Lemma 4.2. *The* ONE SIDE SIMPLIFICATION *algorithm applied on a bipartite graph B with $P \in \{X_B, Y_B\}$ computes in time $O(|P| \times |E_B|)$, the reduction of B produced by the application of reduction operations using only vertices in P until no such reduction is possible.*

Algorithm 2: One Side Simplification

Input: B a bipartite graph, $P \in \{X_B, Y_B\}$

Output: The closure of B by the reduction operation restricted for vertices in P

```
  begin
    % LI = {u ∈ P | N_B(u) ≠ ∅ and Succ(u) ≠ ∅}
    % ∀u, v ∈ P, △(u, v) = |N_B(u)\N_B(v)|
1   for each u, v ∈ P such that u ≠ v and N_B(u) ≠ ∅ do
      △(u, v) ← d_B(u)
2     for each w ∈ N_B(v) do
        if w ∈ N_B(u) then △(u, v) ← △(u, v) − 1
      if △(u, v) = 0 then
        Add(u, LI)
        Add(v, Succ(u))
3   while Not Empty?(LI)) do
      Choose(u, LI)
4     for each v ∈ Succ(u) do
5       for each w ∈ N_B(u) do
6         Remove({v, w}, E_B)
7         for each t ∈ P do
            if t ∈ N_B(w) then
              △(t, v) ← △(t, v) + 1
              if In?(t, LI) then
                Remove(v, Succ(t))
                if Empty?(Succ(t)) then Remove(t, LI)
            else
              △(v, t) ← △(v, t) − 1
              if △(v, t) = 0 and N_B(v) ≠ ∅ then
                Add(v, LI)
                Add(t, Succ(v))
    Remove(u, LI)
  end
```

Proof of the correctness of One Side Simplification. It can be shown easily that after loop 1, $\triangle(u,v)$, LI, $Succ(u)$ are correctly initialized. In the same way, the correctness of these data at each turn of loop 4 is obvious. □

Proof of the complexity of One Side Simplification.

- *The data structures and their complexity.* The sets LI and $Succ$ are two arrays of boolean variables; two integer variables are used to maintain the number of vertices in each set. They are updated by the `Remove` and `Add` operations. A matrix of $|P|^2$ integer variables is used to store $\triangle(u,v)$ for each $u, v \in P$. These data structures need $O(|P|^2)$ space to be stored; they allow to perform the `Choose` operation in time $O(|P|)$ and, the `Remove`, `Add` and `In?` operations in constant time.

- *Total time complexity*. For a vertex u, the neighborhood of each v different from u is visited in loop 2. So, for any u, the total cost for the initialization of $\triangle(u,v)$ for all v different from u is $O(|E_B|)$. Then, the global cost of loop 1 is $O(|P| \times |E_B|)$.

  The total cost of 6 and 7 is $O(|P| \times |\mathit{removed\ edges}|)$, and this is bounded by $O(|P| \times |E_B|)$. At each turn of loop 3, visiting $N_B(u)$ in loop 5 takes $O(d_B(u) \times |\mathit{Succ}(u)|)$ time, that is $O(|\mathit{removed\ edges}|)$ time; so the total cost is bounded by $O(|E_B|)$. Visiting $\mathit{Succ}(u)$ in loop 4 can be done in $O(|P|)$ and as loop 3 is repeated at most $|E_B|$ times, the global cost of loop 3 is $O(|P| \times |E_B|)$. □

Theorem 4.1. *The* SIMPLIFICATION *algorithm applied on a bipartite graph B computes B^s in $O(n \times m)$ time and $O(n^2)$ space.*

5. Computing a minimum cover of a simplified domino-free graph

Property 5.1. *Let B be a bipartite simplified graph. Then $\mathscr{S}(B) = \mathscr{S}_M(B)$.*

Proof. Trivial as, by Definition 4.3, any star of B is a maximal biclique. □

Definition 5.1. Let B a bipartite graph. We denote by $G(B)$ the transitive reduction (Hasse diagram) of $\mathrm{Gal}(B) = (\mathscr{K}_M^*(B), <)$, the Galois lattice of B.

Theorem 5.1. *Let B be a bipartite domino-free simplified graph. There is a bijection between the edges of B and the maximal paths in $G(B)$.*

Proof. Let e be an edge of B. By Theorem 3.3, the bicliques of $\mathscr{K}_M(B,e)$ are pairwise comparable and then form a path of $G(B)$. By Property 5.1, the ends of this path are stars of B, then we obtain a maximal path by adding the vertices $\top$ and $\bot$ to the path.

Inversely, let $(\bot, K_1, \ldots, K_u, \top)$ be a maximal path in $G(B)$. By Property 5.1, K_1 and K_u are stars of B, and we have $K_1 < K_u$. Then $K_1 = (N_B(y), \{y\})$ and $K_u = (\{x\}, N_B(x))$ with $x \in N_B(y)$. To this path, the associated edge of B is then $\{x, y\}$. □

Corollary 5.1. *Let B be a bipartite domino-free simplified graph. Then a subset $\mathscr{C}$ of $\mathscr{K}_M(B)$ is a minimum biclique cover of B if and only if $\mathscr{C}$ is a minimum $\bot - \top$ separator of $G(B)$.*

Proof. A $\bot - \top$ minimum separator $G(B)$ is a minimum transversal of $\bot - \top$ paths in $G(B)$, then the corollary follows by Theorem 5.1. □

Corollary 5.2. *If B is a bipartite simplified domino-free graph with n vertices and m edges, then $G(B)$ has at most $n + m$ edges and at most $n + m$ vertices.*

Proof. Let $K=(X,Y)$ be a non star maximal biclique of B; then $|X|\geqslant 2$ and $|Y|\geqslant 2$. As any star is a maximal biclique of B, K has at least two successors and two predecessors in $G(B)$. If we suppress K of $G(B)$ and add all the edges between the predecessors and the successors of K in $G(B)$, the number of maximal paths does not change, and the number of edges does not decrease. Then the number of edges in $G(B)\backslash\{\top,\bot\}$ is less than the number of maximal paths which is m by Theorem 5.1; so the total number of edges in $G(B)$ is less than $n+m$. By connexity of $G(B)$, the number of vertices is less than $m+n$. □

Corollary 5.3. *Let B a bipartite simplified domino-free graph; given $G(B)$ the transitive reduction of its Galois lattice, a minimum biclique cover of B can be computed in $O(n\times m)$ time.*

Proof. By Corollary 5.1, we only need to compute a minimum $\bot-\top$ separator of $G(B)$. This can be done by using networks flows techniques (cf. [1]) in $O(|A|\sqrt{|V|})$ where $|V|$ is the number of vertices of $G(B)$, and $|A|$ the number of its edges. The result then follows from Corollary 5.2. □

As we have noted in Section 3.2 that $G(B)$ can be computed in polynomial time on bipartite domino-free graphs, this result is sufficient to establish that *s-dim* can be computed by a polynomial time algorithm. Nevertheless, to achieve an overall $O(n\times m)$ time complexity, it is necessary to compute $G(B)$ in $O(n\times m)$ time or less. So, the two next sections provide $O(n\times m)$ time algorithms to check the domino-free property and to compute the Galois lattice of bipartite domino-free graphs. Moreover, these algorithms deal with general bipartite domino-free graphs (not necessarily simplified ones).

6. Checking the domino-free property

6.1. The domino-free property as a local property

Definition 6.1 (*Neighbor order property* (n.o.p.)). Let B be a bipartite graph. To any vertex x of X_B is associated $B_x=B(X_{B_x},Y_{B_x})$ the subgraph of B induced by $Y_{B_x}=N_B(x)$ and $X_{B_x}=N_B(Y_{B_x})$.

B_x has the *neighbour order property* if and only if $\forall x_1,x_2\in X_{B_x}$ such that $N_{B_x}(x_1)\cap N_{B_x}(x_2)\neq\emptyset$, then $d_{B_x}(x_1)\leqslant d_{B_x}(x_2)\Leftrightarrow N_{B_x}(x_1)\subseteq N_{B_x}(x_2)$.

Theorem 6.1. *A bipartite graph B is domino-free if and only if $\forall x\in X_B$, B_x has the n.o.p.*

Proof. ($\Rightarrow$) Suppose $\exists x\in X_{B_x}$ such that B_x does not have the n.o.p. Then $\exists x_1,x_2\in X_{B_x}$ such that x_1 and x_2 have a common neighbour y, $d_{B_x}(x_1)\leqslant d_{B_x}(x_2)$ and $N_{B_x}(x_1)\nsubseteq N_{B_x}(x_2)$.

By definition of B_x, $N_{B_x}(x) = Y_{B_x}$. If $x = x_1$, we have $d_{B_x}(x_1) = d_{B_x}(x_2) = |Y_{B_x}|$ then $N_{B_x}(x_1) = N_{B_x}(x_2)$, which contradicts the hypothesis. If $x = x_2$, $N_{B_x}(x_2) = Y_{B_x}$ then $N_{B_x}(x_1) \subseteq N_{B_x}(x_2)$ and again a contradiction. The last case occurs when x, x_1, x_2 are all distinct; we have then the three edges: $\{x, y\}, \{x_1, y\}, \{x_2, y\} \in E_{B_x}$. As $N_{B_x}(x_1) \not\subseteq N_{B_x}(x_2)$, $\exists y_1 \in N_{B_x}(x_1) \backslash N_{B_x}(x_2)$, so $\{x_1, y_1\}, \{x, y_1\} \in E_{B_x}, \{x_2, y_1\} \notin E_{B_x}$. Moreover $d_{B_x}(x_1) \leqslant d_{B_x}(x_2)$, so $N_{B_x}(x_2) \not\subseteq N_{B_x}(x_1)$; this implies $\exists y_2 \in N_{B_x}(x_2) \backslash N_{B_x}(x_1)$ and we have $\{x, y_2\}$, $\{x_2, y_2\} \in E_{B_x}$ and $\{x_1, y_2\} \notin E_{B_x}$. Therefore, the subgraph of B_x induced by $\{x, x_1, x_2, y, y_1, y_2\}$ is the domino of Fig. 2.

($\Leftarrow$) Suppose that B is not domino-free. Then there exists a domino induced by $\{x, x_1, x_2, y, y_1, y_2\}$, where $\{x, y\}$ is the chord of the domino. Without loss of generality, we can suppose that $d_{B_x}(x_1) \leqslant d_{B_x}(x_2)$; but we do not have $N_{B_x}(x_1) \subseteq N_{B_x}(x_2)$, as $y_1 \in N(x_1)$ and $y_1 \notin N(x_2)$. □

Definition 6.2. $\overset{x}{\approx}$ is the equivalence relation on X_{B_x} defined by

$$\forall x_i, x_j \in X_{B_x},\ x_i \overset{x}{\approx} x_j \ \Leftrightarrow\ N_{B_x}(x_i) = N_{B_x}(x_j).$$

We denote by $\overline{B_x}$ the quotient graph of B_x by $\overset{x}{\approx}$ and, for any x_i in X_{B_x}, we denote by $\overline{x_i}$ the equivalence class of x_i.

$\overset{x}{\prec}$ is the order on $X_{\overline{B_x}}$ defined by

$$\forall \overline{x_i}, \overline{x_j} \in X_{\overline{B_x}},\ \overline{x_i} \overset{x}{\prec} \overline{x_j} \ \Leftrightarrow\ N_{\overline{B_x}}(\overline{x_i}) \subset N_{\overline{B_x}}(\overline{x_j}).$$

Property 6.1. *B_x has the n.o.p. if and only if $\forall y \in Y_{B_x}$, $(N_{\overline{B_x}}(y), \overset{x}{\prec})$ is a total order.*

Proof. Trivial by Definitions 6.1 and 6.2. □

Let us make the link between Theorems 6.1 and 3.3.

Definition 6.3. Let B be a bipartite graph and $x \in X_B$. $\mathscr{K}_M(B, x) = \{(X_i, Y_i) \in \mathscr{K}_M(B) \mid x \in X_i\}$ is the set of all maximal bicliques of B containing x.

Definition 6.4. Let B be a bipartite graph and $\overline{x_k} \in X_{\overline{B_x}}$. $\mathscr{BC}_x(\overline{x_k})$ is the biclique defined by

$$\mathscr{BC}_x(\overline{x_k}) = (\{x_i \in \overline{x_k}\} \cup \{x_i \in \overline{x_j} / \overline{x_k} \overset{x}{\prec} \overline{x_j}\},\ N_{\overline{B_x}}(\overline{x_k})).$$

Property 6.2. *Let B be a bipartite domino-free graph and $x \in X_B$. Then $(X_{\overline{B_x}}, \overset{x}{\prec})$ is isomorphic to $(\mathscr{K}_M(B, x), <)$.*

Proof. We have obviously $\mathscr{K}_M(B, x) = \mathscr{K}_M(B_x)$.

Let us show that $\mathscr{BC}_x$ is a bijection from $X_{\overline{B_x}}$ to $\mathscr{K}_M(B_x)$.

- $\forall \overline{x_i}, \overline{x_j} \in X_{\overline{B_x}}$, and $\overline{x_i} \neq \overline{x_j}$, $\mathscr{BC}_x(\overline{x_i})$, $\mathscr{BC}_x(\overline{x_j}) \in \mathscr{K}_M(B_x)$ and $\mathscr{BC}_x(\overline{x_i}) \neq \mathscr{BC}_x(\overline{x_j})$. Trivial by Definition 6.2.

- $\forall K_i \in \mathscr{K}_M(B_x)$, $\exists \overline{x_i} \in X_{\overline{B_x}}$ s.t. $\mathscr{BC}_x(\overline{x_i}) = K_i$.
  Let $y \in Y_i$. As K_i is a biclique of B_x, y is a neighbour of any $x_j \in X_i$. So, by Property 6.1, the set $S = \{\overline{x_j}/x_j \in X_i\}$ is totally ordered by $\overset{x}{\prec}$. Then, given $\overline{x_i} = Min(S, \overset{x}{\prec})$, we have $\mathscr{BC}_x(\overline{x_i}) = K_i$.
  Let us show that $\forall \overline{x_i}, \overline{x_j} \in X_{\overline{B_x}}$, $\overline{x_i} \overset{x}{\prec} \overline{x_j} \Leftrightarrow \mathscr{BC}_x(\overline{x_i}) < \mathscr{BC}_x(\overline{x_j})$.
  This can be obtained directly from Definition 6.2 which provides $\overline{x_i} \overset{x}{\prec} \overline{x_j} \Leftrightarrow N_{\overline{B_x}}(\overline{x_i}) \subset N_{\overline{B_x}}(\overline{x_j})$. Given $\mathscr{BC}_x(\overline{x_i}) = K_i$ and $\mathscr{BC}_x(\overline{x_j}) = K_j$, we have by Definition 6.4 of $\mathscr{BC}_x$, $Y_i = N_{\overline{B_x}}(\overline{x_i})$ and $Y_j = N_{\overline{B_x}}(\overline{x_j})$ and then, $Y_i \subset Y_j \Leftrightarrow K_i < K_j$. □

Note: As $N_{\overline{B_x}}(y)$ is merely the subset of $X_{\overline{B_x}}$ whose elements have y as a common neighbour, $\{\mathscr{BC}_x(\overline{x_i}),\ \overline{x_i} \in N_{\overline{B_x}}(y)\}$ is simply $\mathscr{K}_M(B, \{x, y\})$ (Property 6.2). So the Property 6.1 is only a restatement of the Theorem 3.3.

Corollary 6.1. *Let B be a bipartite domino-free graph. The number of maximal bicliques of B is bounded by n^2.* [1]

Proof. From Property 6.2, $|\mathscr{K}_M(B,x)| = |X_{\overline{B_x}}| \leqslant n$, so $|\mathscr{K}_M(B)| < n^2$. □

Definition 6.5. Let B be a bipartite graph and $x \in X_B$.
Let $G(B,x)$ be the transitive reduction of $(X_{\overline{B_x}}, \overset{x}{\prec})$ and, for simplicity, let us denote it by $(X_{\overline{B_x}}, \Gamma_x^-)$ with $\Gamma_x^-(\overline{x_i})$, the set of immediate predecessors of $\overline{x_i}$ in $G(B,x)$.

Corollary 6.2. *Let B be a bipartite domino-free graph and $x \in X_B$. $G(B,x)$ is a tree.*

Proof. Trivial by Corollary 3.1 and Property 6.2. □

6.2. Checking the domino-free property in O($n \times m$)

Lemma 6.1. Local checking *computes Γ_x^- in $O(|X_{\overline{B_x}}| + |Y_{\overline{B_x}}| + |E_{\overline{B_x}}|)$ time if B_x has the n.o.p. and aborts otherwise.*

Proof of the correctness of Local checking. If Local checking succeeds, it follows from Property 6.1 that B_x has the n.o.p. It is obvious then, that at each turn of loop 3, the computed function Γ_x^- for vertices in $N_{\overline{B_x}}(y)$ is correct. Moreover, by Definition 6.2 of $\overset{x}{\prec}$, comparable vertices in $X_{\overline{B_x}}$ share at least a common neighbour y. So, we have $\bigcup_{y \in Y_{\overline{B_x}}} (N_{\overline{B_x}}(y), \overset{x}{\prec}) = (X_{\overline{B_x}}, \overset{x}{\prec})$ and then, Local checking returns a correct Γ_x^-.

If it aborts at line 4, there is $\overline{x_j}$ such that $\overline{x_j}$ has two immediate successors in $G(B,x)$, which contradicts Corollary 6.2; if it aborts at line 6, there are $y \in Y_{\overline{B_x}}$ and

[1] In fact, it can be shown that $|\mathscr{K}_M(B)| \leqslant m$; the proof is rather tedious, and can be found in [2].

$\overline{x_i}, \overline{x_j} \in N_{\overline{B_x}}(y)$ such that $d_{\overline{B_x}}(\overline{x_i}) < d_{\overline{B_x}}(\overline{x_j})$ and $N_{\overline{B_x}}(\overline{x_i}) \not\subset N_{\overline{B_x}}(\overline{x_j})$, so, following Definition 6.1, B_x has not the n.o.p. □

Proof of the complexity of Local checking. *Space complexity*: T_x is stored as an array of lists so requires $O(|X_{\overline{B_x}}| + |Y_{\overline{B_x}}|)$ space; similarly, V is stored as an array of lists and so requires $O(|Y_{\overline{B_x}}| + |E_{\overline{B_x}}|)$ space; space needed to store *Succ* is $O(|X_{\overline{B_x}}|)$.

Time complexity: loop 1 is in time $O(|X_{\overline{B_x}}|)$. Loop 2 needs $O(|Y_{\overline{B_x}}| + |E_{\overline{B_x}}|)$. Loop 3 is in time $O(|E_{\overline{B_x}}|)$. Loop 5 can be done in $O(|E_{\overline{B_x}}|)$ by searching breadth-first Γ_x^-. So the total time complexity is $O(|X_{\overline{B_x}}| + |Y_{\overline{B_x}}| + |E_{\overline{B_x}}|)$. □

Lemma 6.2. *Let B be a bipartite graph and $x \in X_B$. Given B_x, $\overline{B_x}$ can be computed in $O(|X_{B_x}| + |Y_{B_x}| + |E_{B_x}|)$ time.*

Proof. The computation of $\overline{B_x}$ from the given B_x amounts to compute the relation $\overset{x}{\approx}$ on X_{B_x}. This can be done in $O(|X_{B_x}| + |Y_{B_x}| + |E_{B_x}|)$ by successively refining for each vertex in Y_{B_x}, a partition of X_{B_x} initially having one part. □

Theorem 6.2. *Checking the domino-free property for a given bipartite graph B can be done in $O(n \times m)$ time.*

Proof. Simply by checking the n.o.p. of each B_x using Local checking (cf. Theorem 6.1). The complexity directly follows from Lemmas 6.2 and 6.1. □

Algorithm 3. Local checking
Input: $\overline{B_x}$
Output: $G(B,x) = (X_{\overline{B_x}}, \Gamma_x^-)$ iff B_x has the n.o.p.

begin
% $T_x(k)$ is the set of vertices $\overline{x_i}$ s.t. $d_{\overline{B_x}}(\overline{x_i}) = k$, for k in $1..|Y_{\overline{B_x}}|$
% $V(y)$ is $N_{\overline{B_x}}(y)$ sorted by ascending degrees in $\overline{B_x}$ for $y \in Y_{\overline{B_x}}$
% for $\overline{x_i}$ in $V(y), Pred_{V(y)}(\overline{x_i})$ is the predecessor of $\overline{x_i}$ *in* $V(y)$
% $Succ(\overline{x_i})$ is the immediate successor of $\overline{x_i}$ in $G(B,x)$ for $\overline{x_i} \in X_{\overline{B_x}}$

1 **for** *each* $\overline{x_i} \in X_{\overline{B_x}}$ **do**
  $\Gamma_x^-(\overline{x_i}) \leftarrow nil$
  $Succ(\overline{x_i}) \leftarrow nil$
  Add($\overline{x_i}$, $T_x(d_{\overline{B_x}}(\overline{x_i}))$)

2 **for** *each* k *in* $1..|Y_{\overline{B_x}}|$ **do**
  for *each* $\overline{x_i}$ *in* $T_x(k)$ **do**
    for *each* y *in* $N_{\overline{B_x}}(\overline{x_i})$ **do**
      AddEnd($\overline{x_i}$, $V(y)$)

3 **for** *each* $y \in Y_{\overline{B_x}}$ **do**
  for *each* $\overline{x_i} \in V(y)$ **do**

if $Succ(Pred_{V(y)}(\overline{x_i})) = nil$ **then**
  Add($Pred_{V(y)}(\overline{x_i})$, $\Gamma_x^-(\overline{x_i})$)
  $Succ(Pred_{V(y)}(\overline{x_i})) \leftarrow \overline{x_i}$
else
  if $Succ(Pred_{V(y)}(\overline{x_i})) \neq \overline{x_i}$ **then**
4   **return** "*B_x is not domino-free*"
5 **for** *each* $\overline{x_i}, \overline{x_j}$ *such that* $\overline{x_j} \in \Gamma_x^-(\overline{x_i})$ **do**
  if $N_{\overline{B_x}}(\overline{x_j}) \not\subset N_{\overline{B_x}}(\overline{x_i})$ **then**
6   **return** "*B_x is not domino-free*"
return $G(B,x)$
end

7. Computing the Galois lattice of bipartite domino-free graphs in $O(n \times m)$ time

Lemma 7.1. *Let B be a bipartite domino-free graph and $x \in X_B$. Given $\overline{B_x}$ and Γ_x^-, $\mathscr{BC}_x$ can be computed in $O(|X_{B_x}| + |Y_{B_x}| + |E_{B_x}|)$ time.*

Proof. The computation of $\mathscr{BC}_x$ requires only to visit $G(B,x)$ in a depth-first search manner, computing $N_{\overline{B_x}}(\overline{x_i})$ for each $\overline{x_i}$ of $G(B,x)$. □

By Property 6.2 and Lemma 7.1, the algorithm LOCAL CHECKING allows us to compute $(\mathscr{K}_M(B,x), \Gamma_x^-)$, the transitive reduction of $(\mathscr{K}_M(B,x), <)$. So in the following, for simplicity sake, $G(B,x)$ denotes as well $(X_{\overline{B_x}}, \Gamma_x^-)$ as $(\mathscr{K}_M(B,x), \Gamma_x^-)$.

Definition 7.1. Let B be a bipartite graph, $V \subseteq X_B$

- $\mathscr{K}_M(B,V) = \bigcup_{x \in V} \mathscr{K}_M(B,x)$ is the set of all maximal bicliques of B containing at least a vertex of V.
- $G(B,V) = (\mathscr{K}_M(B,V), \Gamma_V^-)$ is the subgraph of the transitive reduction of $(\mathscr{K}_M(B), <)$ induced by $\mathscr{K}_M(B,V)$.

Definition 7.2. Let B be a bipartite graph, $V \subseteq X_B$ and $e \in E_B$.

- If there exists some biclique of $\mathscr{K}_M(B,V)$ covering e, then we denote by $\mathscr{G}_V(e) = K_i \in \mathscr{K}_M(B,V)$ the greatest of these bicliques and take $|\mathscr{G}_V(e)| = |Y_i|$.
- Otherwise, we take the following understanding: $\mathscr{G}_V(e) = \perp$; $|\mathscr{G}_V(e)| = 0$.

Lemma 7.2. *Let B be a bipartite domino-free graph, $V \subset X_B$, $x \in X_B \mid V$, $K_1 \in \mathscr{K}_M(B,x)$ and $e \in E_{K_1}$.*

$$K_1 \in \mathscr{K}_M(B,V) \Leftrightarrow |Y_1| \leqslant |\mathscr{G}_V(e)|.$$

Proof. ($\Rightarrow$) Trivial from Definition 7.2.

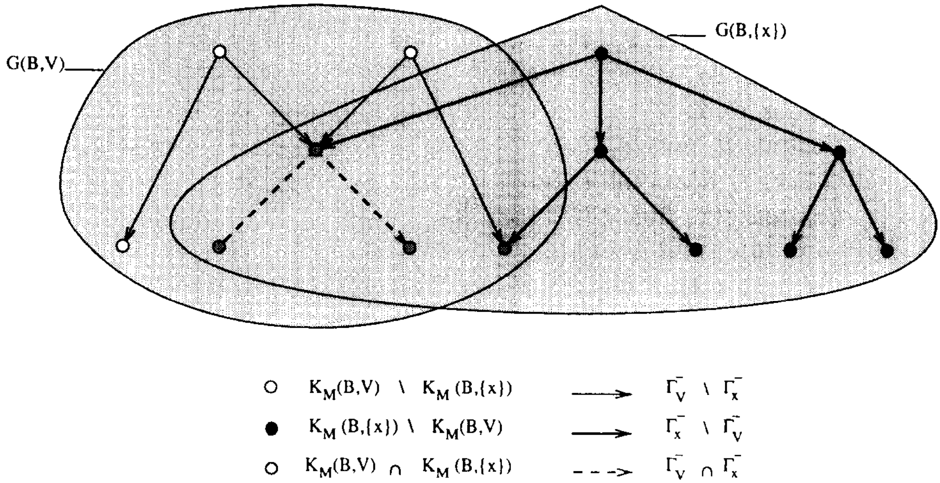

Fig. 6. A step of the Galois algorithm.

($\Leftarrow$) If $|Y_1| \leqslant |\mathscr{G}_V(e)|$ then, as they both cover e, we can deduce from Property 3.1 (a): $\mathscr{G}_V(e) = K1$ or $\mathscr{G}_V(e) > K1$. Moreover, by Definition 7.2, there is $x' \in V$ such that $\mathscr{G}_V(e) \in \mathscr{K}_M(B,x')$. We have then $x' \in X_1$ (a) and finally, $K_1 \in \mathscr{K}_M(B,x')$. □

In other words, for $K_1 \in \mathscr{K}_M(B,x)$, either $K_1 \in \mathscr{K}_M(B,V)$ and all edges e of K_1 are such that $|Y_1| \leqslant |\mathscr{G}_V(e)|$, or else $K_1 \notin \mathscr{K}_M(B,V)$ and all edges e of K_1 are such that $|Y_1| > |\mathscr{G}_V(e)|$.

Lemma 7.3. *Let B be a bipartite domino-free graph. Then*:
(a) *For $x \in X_B$, $\forall e = \{x_i, y_j\} \in E_{B_x}$, $\mathscr{G}_{\{x\}}e = \mathscr{BC}_x(\overline{x_i})$.*
(b) *For $V \subset X_B$, $x \in X_B \setminus V$,*

$$\forall e \in E_{B_x} \text{ s.t. } \mathscr{G}_{\{x\}}(e) \in \mathscr{K}_M(B,x) \setminus \mathscr{K}_M(B,V),\ \mathscr{G}_{\{x\}}(e) > \mathscr{G}_V(e).$$

Proof. (a) follows directly from Definition 6.4 and Property 6.2.
(b) Follows directly from Lemma 7.2. □

The following Galois algorithm computes incrementally the transitive reduction of the Galois lattice of B: at each step, the Link algorithm is used to extend the order associated with the bicliques containing at least one vertex in $V \subseteq X_B$ with the bicliques containing a new vertex x ($x \in X_B \setminus V$) and not already found.

Algorithm 4. Galois
Input: B a domino-free bipartite graph
Output: $G(B)$

`Var(`V`)`

begin
  $V \leftarrow \emptyset$; Initialize $\mathscr{G}_V$; $\mathscr{K}_M(B,V) \leftarrow \emptyset$; $\Gamma_V^- \leftarrow \emptyset$
  While $V \neq X_B$ **do**
    Choose $x \in X_B \backslash V$
    Compute $\overline{B_x}$ from B_x
    $G(B,x) \leftarrow$ LOCAL CHECKING($\overline{B_x}$)
    Compute $\mathscr{BC}_x$ from $G(B,x)$ and $\overline{B_x}$
      $\mathscr{K}_M(B, V \cup \{x\}) \leftarrow \mathscr{K}_M(B,V)$
      $\Gamma_{V \cup \{x\}}^- \leftarrow \Gamma_V^-$
    LINK($\overline{x}$)
    Add(x, V)
  return $(\mathscr{K}_M(B,X_B), \Gamma_{X_B}^-)$
end

Algorithm 5. LINK
Input: $\overline{x_i} \in X_{\overline{B_x}}$
Output: Add to $\mathscr{K}_M(B, V \cup \{x\})$ all the bicliques of $\mathscr{K}_M(B,x) \backslash \mathscr{K}_M(B,V)$ containing $\overline{x_i}$ and initialize $\Gamma_{V \cup \{x\}}^-$ for each of those bicliques; update $\mathscr{G}_{V \cup \{x\}}$ for each edge covered by an added biclique.

begin
  Choose $e = (x', y)$ such that $x' \in \overline{x_i}$ and $y \in N_{\overline{B_x}}(\overline{x_i})$
  if $d_{\overline{B_x}}(\overline{x_i}) > |\mathscr{G}_V(e)|$ **then**
    Add($\mathscr{BC}_x(\overline{x_i}), \mathscr{K}_M(B, V \cup \{x\})$)
    for *each* $e_i = (x_i, y_i)$ *such that* $x_i \in \overline{x_i}$ *and* $y_i \in N_{\overline{B_x}}(\overline{x_i})$ **do**
      $\mathscr{G}_{V \cup \{x\}}(e_i) \leftarrow \mathscr{BC}_x(\overline{x_i})$
    for *each* $\overline{x_j} \in \Gamma_x^-(\overline{x_i})$ **do**
      Add(LINK($\overline{x_j}$), $\Gamma_{V \cup \{x\}}^-(\mathscr{BC}_x(\overline{x_i}))$)
    return $\mathscr{BC}_x(\overline{x_i})$
  else
    return $\mathscr{G}_V(e)$
end

Lemma 7.4. LINK *computes* $G(B, V \cup \{x\})$ *using* $G(B,V)$ *and* $G(B,x)$ *in* $O(|X_{B_x}| + |Y_{B_x}| + |E_{B_x}|)$ *time.*

Proof. The correctness of LINK follows directly from Lemma 7.2 and from Property 6.2. Its complexity follows from Corollary 6.2. □

Theorem 7.1. *The* GALOIS *algorithm computes the transitive reduction of the Galois lattice of a bipartite domino-free graph* B *in* $O(n \times m)$ *time.*

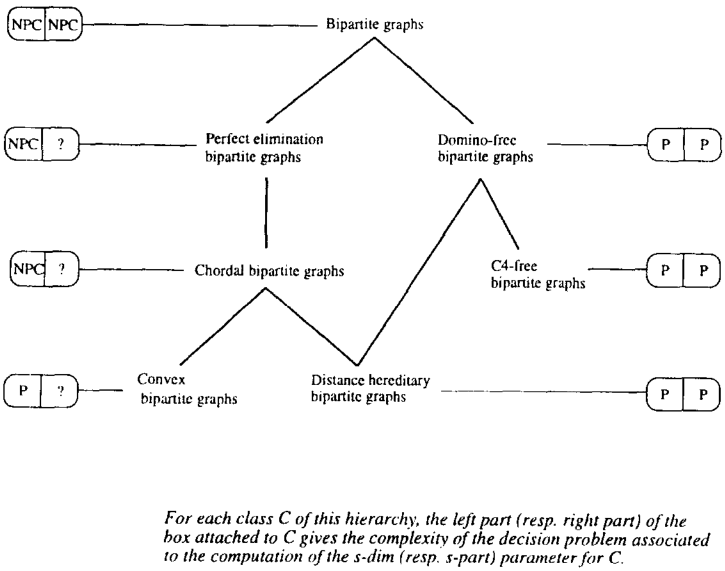

Fig. 7. Complexity of the computation of the *s-dim* and *s-part* parameters.

8. Conclusion and further work

In this paper, we give a new class of bipartite graphs, for which the *s-dim* and *s-part* parameters can be computed polynomially in $O(n \times m)$ time. Incidentally, we show that, for this class, the number of maximal bicliques is bounded by n^2 and that the associated Galois lattice can be computed in $O(n \times m)$ time.

A remaining open question is to better define the border between *P* and *NPC* in regards to the parameters *s-dim* and *s-part*. In particular, there are some others polynomial classes for *s-dim* [12] and *s-part* [11], but they are only defined by closure of some classes with some composition operations. It would be interesting to classify them in the above hierarchy.

Another question is to know whether there are other problems already known in *P* for bipartite C4-free graphs and bipartite distance hereditary graphs which are also in *P* for bipartite domino-free graphs. Connections of these results with some problems of partial order theory deserve a further study, as computing *s-dim* for a bipartite graph is polynomially related to the computing of the dim_2 parameter of a partial order.

References

[1] R. Ahuja, T.L. Magnanti, J.B. Orlin, Network Flows, Theory, Algorithms and Applications, Prentice-Hall, Englewood Cliffs, NJ, 1993.

[2] J. Amilhastre, M.C. Vilarem, P. Janssen, Complexity of minimum biclique cover and decomposition for a class of bipartite graphs, Technical Report 96035, LIRMM, 1995.

[3] J.P. Bordat, Calcul pratique du treillis de Galois d'une correspondance, Math. Sci. Hum. (96) (1986) 31–47.
[4] A. Brandstädt, Special graph classes – a survey, Technical Report SM-DU-199, Schriftenreihe des Fachbereichs Mathematik, 1993.
[5] P.C. Fishburn, P.L. Hammer, Bipartite dimensions and bipartite degrees of graphs, Technical Report 76, DIMACS, June 1993. Available at http://dimacs.rutgers.edu/TechnicalReports.
[6] D.S. Franzblau, D.J. Kleitman, An algorithm for covering polygons with rectangles, Inform. Control 63 (1984) 164–189.
[7] V. Froidure, Rangs des relations binaires, semigroupes de relations non ambigues, Ph.D. Thesis, Univ. Paris VI, June 1995.
[8] M. Habib, L. Nourine, A new lattice-based heuristic for taxonomy encoding, Private communication, 1996.
[9] Tao Jiang, B. Ravikumar, Minimal NFA Problems are hard, SIAM J. Comput. 22 (1993) 1117–1141.
[10] T. Kloks, D. Kratsch, Computing a perfect edge without vertex elimination ordering of a chordal bipartite graph, Inform. Process. Lett. 55 (1995) 11–16.
[11] T. Kratzke, B. Reznick, D. West, Eigensharp Graphs: Decomposition into complete bipartite subgraphs, Trans. Amer. Math. Soc. 308 (2) (1988) 637–653.
[12] A. Lubiw, The boolean basis problem and how to cover some polygons by rectangles, SIAM J. Discrete Math. 3 (1) (1990) 98–115.
[13] A. Lubiw, A weighted min-max relation for intervals, J. Combin. Theory 53 (2) (1991) 151–172.
[14] H. Müller, Alternating cycle-free matchings, Order (7) (1990) 11–21.
[15] H. Müller, On edge perfectness and classes of bipartite graphs, Discrete Math. (149) (1996) 159–187.
[16] D.S. Nau, G. Markowsky, M.A. Woodbury, D.B. Amos, A mathematical analysis of human leukocyte antigen serology, Math. Biosci. 40 (1978) 243–270.
[17] J. Orlin, Containment in Graph Theory: covering graphs with cliques, Nederl. Akad. Wetensch. Indag. Math. 39 (1977) 211–218.
[18] R. Wille, Restructuring lattice theory: an approach based on hierarchies of contexts, in: I. Rival (Ed.), Ordered Sets, NATO ASI, vol. 83, Reidel, Dordrecht, Holland, 1982, pp. 445–470.