1 Introduction

This model computes the prices of interest rate derivative securities. The model is based off risk-neutral interest rates following a Hull-White model which can fit the initial yield curve exactly. The real-world interest rate dynamics are assumed to follow a Vasicek (time-homogeneous Hull-White) process for simulation purposes. This framework is chosen for two reasons: it is efficient in pricing a wide range of interest rate options and it provides a suitable first approximation to the risk that may be incurred from movements in interest rates. Models that provide a better fit to market data (eg, the LIBOR Market Model) are not as flexible in pricing a variety of options. Models that generalize to multi-dimensional processes (eg, multi-dimensional Hull-White) are more difficult to estimate and are often less computationally efficient for pricing.

Inputs to each pricing model include a function representing the initial yield curve and initial forward rate curve.

2 Assumptions

Like all pricing models, the assumption of NFLVR (No Free Lunch With Vanishing Risk) is used. This assumption allows the leveraging of the First Fundamental Theorem of Asset Pricing for pricing purposes (ie, all asset prices are martingales when denominated by another asset under a suitable measure and can be written as an expectation). It is assumed that there exists an asset M_t which satisfies $dM_t = r_t M_t dt$ (ie, instantaneously riskless). Solving the ODE gives $M_t = M_0 e^{\int_0^t r_s ds}$.

The variable r_t is assumed to follow a Vasicek (Ornstein Uhlenbeck) process:

$$dr_t = a(b - r_t)dt + \sigma dW_t$$

Here dW_t is an increment of Brownian Motion, a is the speed of mean reversion, b is the long run expected value of r_t, and σ is the volatility of the process. This process is chosen for its simplicity and rough approximation of the actual interest rate process.

By Girsonav's theorem, $d\tilde{W}_t = dW_t - \frac{a(\theta(t)-b)}{\sigma}dt$ is a Brownian Motion under a suitable measure. Substituting into the equation for the short rate,

$$dr_t = a(b - r_t)dt + \sigma\left(d\tilde{W}_t + \frac{a(\theta(t) - b)}{\sigma}dt\right) = a(\theta(t) - r)dt + \sigma d\tilde{W}_t$$

Since this measure change is applicable for $\sigma > 0$, the model does not allow arbitrage and can be used for pricing interest rate securities. In particular, the time dependent parameter θ can be calibrated to market data (the yield curve)

and then used to price interest rate securities.

By the Fundamental Theorem of Asset Pricing, the price of a zero coupon bond with par one is

$$\begin{aligned} B(t,T) &= M_t \tilde{\mathbb{E}}\left[\frac{1}{M_T}|\mathcal{F}_t\right] \\ &= \tilde{\mathbb{E}}\left[e^{-\int_t^T r_s ds}|\mathcal{F}_t\right] \end{aligned}$$

The final assumption is that the 7 day LIBOR rate is a reasonable proxy for the short rate r_t and that the LIBOR rates (implied or actual) from 1 month to 30 years are the yields on zero coupon bonds. From here on, $M_t/M_T = e^{-\int_t^T r_s ds}$ will be denoted $D(t,T)$.

3 Pricing Modules

The following "generic" functions are available: tree (lattice) methods (3.8.1), analytic Black Scholes formula (3.2), and Jamshidian's trick for finding options on a portfolio (3.7.1). From these generic functions, the fixed income derivatives can be priced.

3.1 Bond Price

Using the bootstrapped yield curve, any bond can be priced as of the current date. The cash flows are discounted at the zero coupon rate and summed. However, this does not facilitate the pricing of bonds at some future date. Hull and White showed that the price of a zero-coupon bond at a date $t > 0$ is

$$B(t,T) = \frac{B(0,T)}{B(0,t)} e^{A(t,T)F(t) - \frac{\sigma^2}{4a}A^2(t,T)\left(1-e^{-2at}\right) - A(t,T)r_t}$$

Where $A(t,T) = \frac{1-e^{-a(T-t)}}{a}$ and $F(t) = -\frac{\partial \log(B(0,t))}{\partial t}$ is the instantaneous forward rate. The instantaneous forward rate is trivially computed from the spline. The time-dependent parameter $\theta(t)$ is related to $F(t)$ as follows:

$$\theta(t) = \frac{1}{a}\frac{\partial F}{\partial t} + F(t) + \frac{\sigma^2}{2a^2}\left(1 - e^{-2at}\right)$$

3.2 Black Scholes

Given a discount factor $P(t,T)$ and an underlying asset S_T which is log-normally distributed, a vanilla option on S_T with strike k with maturity T and cumulative volatility $\sigma\sqrt{T}$ is given by the Black Scholes formula. In particular, for a call option,

$$C(S_t; k, t, T\sigma) = S_t \mathcal{N}(d_1) - kP(t,T)\mathcal{N}(d_1 - \sigma\sqrt{T})$$

Where $d_1 = \log\left(\frac{S_t}{kP(t,T)}\right)/\sigma\sqrt{T} + \frac{1}{2}\sigma\sqrt{T}$.

In a fixed income setting, the underlying asset is a bond denominated by a bond of same tenor as the option. The discount factor and the asset are hence both bonds, and the "σ" is equal to the square root of the integral of the squared difference between the volatilities of the bonds; see (3.2.2).

3.2.1 Forward Measure

The forward measure can help facilitate pricing of caplets and options on bonds. Under this measure, the dynamics of a bond $B(T, T+\delta)$ are

$$d\frac{B(t,T+\delta)}{B(t,T)}$$

$$= \left(\sigma(t,T+\delta) - \sigma(t,T)\right)\frac{B(t,T+\delta)}{B(t,T)}d\hat{W}_t$$

Where $\sigma(t,T)$ is the volatility of $B(t,T)$.

3.2.2 Bond volatility

In the Hull-White model, the volatility of the bond price is deterministic: $\sigma(t,T) = \frac{\sigma}{a}(1 - e^{-a(T-t)})$

$$\implies \sigma(t,T+\delta) - \sigma(t,T) = \frac{\sigma}{a}e^{-a(T-t)}\left(1 - e^{-a\delta}\right)$$

Hence option pricing on the bond reduces to the Black Scholes formula with

$$\sigma_{BS} = \sqrt{\frac{1}{T}\int_0^T \left(\frac{\sigma}{a}e^{-a(T-t)}\left(1 - e^{-a\delta}\right)\right)^2 dt}$$

$$\sigma_{BS} = \frac{\sigma}{a}\left(1 - e^{-a\delta}\right)\sqrt{\frac{1}{T}\int_0^T e^{-2a(T-t)}dt}$$

$$\sigma_{BS} = \frac{\sigma}{a}\left(1 - e^{-a\delta}\right)\sqrt{\frac{1 - e^{-2aT}}{2aT}}$$

Note that the volatility depends only on a and σ and hence these parameters can be calibrated to the cap market.

3.3 Caplet and Bond option equivalency

The value of a caplet is

$$\delta\tilde{\mathbb{E}}\left[D(t,T+\delta)\left(\frac{1 - B(T,T+\delta)}{\delta B(T,T+\delta)} - k\right)^+ |\mathcal{F}_t\right]$$

$$= \delta\tilde{\mathbb{E}}\left[D(t,T+\delta)\left(\frac{1}{\delta B(T,T+\delta)} - k - \frac{1}{\delta}\right)^{+}|\mathcal{F}_t\right]$$

$$= \delta\tilde{\mathbb{E}}\left[D(t,T)\left(\frac{\tilde{\mathbb{E}}[D(T,T+\delta)|\mathcal{F}_T]}{\delta B(T,T+\delta)} - \tilde{\mathbb{E}}[D(T,T+\delta)|\mathcal{F}_T]\left(k+\frac{1}{\delta}\right)\right)^{+}|\mathcal{F}_t\right]$$

$$= \delta\tilde{\mathbb{E}}\left[D(t,T)\left(\frac{1}{\delta} - B(T,T+\delta)\left(k+\frac{1}{\delta}\right)\right)^{+}|\mathcal{F}_t\right]$$

$$= (\delta k+1)\,\tilde{\mathbb{E}}\left[D(t,T)\left(\frac{1}{\delta k+1} - B(T,T+\delta)\right)^{+}|\mathcal{F}_t\right]$$

$$= (k\delta+1)\,P\left(\frac{1}{\delta k+1}\right)$$

Where $P(\cdot)$ is a put on a bond with strike $\cdot$.

3.4 Caplet pricing

From above, the price of a caplet is the following:

$$c(t,T,\delta;k) = (k\delta+1)\,\tilde{\mathbb{E}}\left[D(t,T)\left(\frac{1}{\delta k+1} - B(T,T+\delta)\right)^{+}|\mathcal{F}_t\right]$$

$$= (k\delta+1)\,B(t,T)\hat{\mathbb{E}}\left[\left(\frac{1}{\delta k+1} - B(T,T+\delta)\right)^{+}|\mathcal{F}_t\right]$$

Where $\hat{E}$ is the measure induced by $B(t,T)$. The forward measure can then be used to price the option.

3.5 Euro Dollar Futures

The future price satisfies

$$\delta f(t,T;\delta) = \tilde{\mathbb{E}}\left[\frac{1}{B(T,T+\delta)}|\mathcal{F}_t\right] - 1$$

In a Hull-White model, the future price is

$$\delta f(t,T;\delta) = \frac{B(t,T)}{B(t,T+\delta)}e^{\gamma(t,T,\delta)} - 1$$

For some deterministic function γ. This function is

$$\int_t^T \sigma_B(s,T+\delta)\left(\sigma_B(s,T+\delta) - \sigma_B(s,T)\right)ds$$

This equals $\frac{\sigma^2}{a^3}\left(1-e^{-a\delta}\right)\left(1-e^{-a(T-t)} - \frac{e^{-a\delta}}{2}\left(1-e^{-2a(T-t)}\right)\right)$.

3.6 Swap

An interest rate swap is an instrument in which two parties exchange rates: typically fixed for floating. This exchange is in arrears, that is, after the time period δ of the floating rate has passed. Letting $B(t,T)$ be the price of a zero coupon bond at time t maturity and time T, the payoff the receiver of floating is the following:

$$\sum_{i=0}^{n} \frac{1 - B(t_i, t_i + \delta)}{B(t_i, t_i + \delta)\delta} - k$$

Where $t_0 = t$, $t_n = T$ and T is the maturity. The price of this swap is

$$\sum_{i=0}^{n} \tilde{\mathbb{E}} \left[D(t_0, t_i + \delta) \left(\frac{1 - B(t_i, t_i + \delta)}{B(t_i, t_i + \delta)\delta} - k \right) | \mathcal{F}_{t_0} \right]$$

By convention, this price is zero at the time of origination. k is termed the swap rate, and is solved by setting the equation equal to zero:

$$\sum_{i=0}^{n} \tilde{\mathbb{E}} \left[D(t_0, t_i + \delta) \left(\frac{1 - B(t_i, t_i + \delta)}{B(t_i, t_i + \delta)\delta} - k \right) | \mathcal{F}_{t_0} \right] = 0$$

$$\sum_{i=0}^{n} \tilde{\mathbb{E}} \left[\frac{D(t_0, t_i + \delta)}{B(t_i, t_i + \delta)} | \mathcal{F}_{t_0} \right] = \sum_{i=0}^{n} \tilde{\mathbb{E}} \left[D(t_0, t_i + \delta) \left(1 + k\delta\right) | \mathcal{F}_{t_0} \right]$$

$$\sum_{i=0}^{n} B(t_0, t_i) = \sum_{i=0}^{n} B(t_0, t_i + \delta) \left(1 + k\delta\right)$$

$$k = \frac{1}{\delta} \left(\frac{\sum_{i=0}^{n} B(t_0, t_i)}{\sum_{i=0}^{n} B(t_0, t_i + \delta)} - 1 \right)$$

t_i is typically chosen as a multiple of δ. In this case, the swap rate can be further simplified:

$$k = \frac{1}{\delta} \left(\frac{1 - B(t_0, t_n + \delta) + \sum_{i=0}^{n} B(t_0, t_i + \delta)}{\sum_{i=0}^{n} B(t_0, t_i + \delta)} - 1 \right)$$

$$= \frac{1}{\delta} \left(\frac{1 - B(t_0, t_n + \delta)}{\sum_{i=0}^{n} B(t_0, t_i + \delta)} \right)$$

3.7 Equivalence of Option on Coupon Bond and European Swaption

A swaption is the right to enter a swap at a future date at a predetermined swap rate. This can be written as

$$\tilde{\mathbb{E}} \left[D(t,T) \left(1 - \sum c_j B(T, T_j) \right)^{+} | \mathcal{F}_t \right]$$

Where $c_j = k\delta$, $j < n$ and $c_n = 1 + k\delta$. Hence a swaption is a put on a coupon bond with coupon rate $k\delta$ and strike 1.

3.7.1 Jamshidian's Trick

Jamshidian pointed out that an option with payoff $(\sum f(r) - k)^+$ is equal to a portfolio of options $\sum (f(r) - f(r_*))^+$ where r_* is the solution to $\sum f(r) - f(r_*) = 0$ and f is a monotonic function. Hence an option on a coupon bond can be written as a series of options on zero coupon bonds. Solving for r_* involves Newton's method for finding the zero's of a function. Valuing an option on a coupon bond or, equivalently, a European swaption can thus be computed using Jamshidian's trick and multiple calls to the pricing function of an option on a zero coupon bond.

3.8 American Swaption

Note that in the Hull-White model that the short rate can be decomposed as $r_t = \phi(t) + x_t$ where $\theta(t) = \frac{\phi'(t)}{a} + \phi(t)$ and $dx_t = -ax_t dt + \sigma dW_t$. As there is no analytic formula for American options, the tree method is used to step backwards in time. The tree is solved in x and then $\phi(t)$ is added to the result for pricing and discounting.

3.8.1 Tree Methods

Consider the SDE $dX = \alpha(X,t)dt + \sigma(X,t)dW_t$. Now consider $f(y)$ such that $f'(y) = \frac{1}{\sigma(y,t)}$. Then

$$df(X) = \frac{1}{\sigma(X,t)}\alpha(X,t)dt + dW_t - \frac{\sigma^2(X,t)\sigma'(X)dt}{2\sigma^2(X,t)}$$

$$= \left(\frac{\alpha(X,t)}{\sigma(X,t)} - \frac{\sigma'(X)}{2}\right)dt + dW_t$$

This dynamic can lead to a recombining tree. Letting $y = f(x)$ then $u_i = y_0 + (m-i)\sqrt{\Delta t}$ and $d_i = y_0 - (m-i)\sqrt{\Delta t}$. The actual value at each node is given by $f^{-1}(u_i)$. The probability of the move is solved by $p * u + (1-p) * d = \left(\frac{\alpha(X,t)}{\sigma(X,t)} - \frac{\sigma'(X)}{2}\right)\Delta t$.

$$p\sqrt{\Delta t} - (1-p)\sqrt{\Delta t} = \left(\frac{\alpha(X,t)}{\sigma(X,t)} - \frac{\sigma'(X)}{2}\right)\Delta t$$

$$2p - 1 = \left(\frac{\alpha(X,t)}{\sigma(X,t)} - \frac{\sigma'(X)}{2}\right)\sqrt{\Delta t}$$

$$p = \frac{1}{2}\left(\frac{\alpha(X,t)}{\sigma(X,t)} - \frac{\sigma'(X)}{2}\right)\sqrt{\Delta t} + \frac{1}{2}$$

4 Appendix

4.1 Calibration

This library assumes that there exists a yield curve and forward yield curve. This section gives some methods for calculating such a curve from yield curve data. Since LIBOR rates extend from 7 days to 1 year, the rates from 1 to 30 years must be extrapolated from LIBOR Swap rates.

4.1.1 Yield Curve calibration

Let $S(t_0, T, \delta) = k$. Assume this rate exists for every $T \in t_m + \delta i$ where $i \in 0, ...N$ where $N = (30 - t_m)/\delta$ and t_m is the last maturity of the LIBOR curve (one year out). Note that this assumption is not restrictive as it is possible to create a spline for the missing dates. From the definition of k,

$$S(t_0, t_m, \delta) = \frac{1}{\delta}\left(\frac{1 - B(t_0, t_m + \delta)}{\sum_{i=0}^{n} B(t_0, t_i + \delta)}\right)$$

Note that we have all the values in this expression except for $B(t_0, t_m + \delta)$. Solving for this value,

$$S(t_0, t_m, \delta)\delta B(t_0, t_m + \delta) + S(t_0, t_m, \delta)\delta \sum_{i=0}^{n-1} B(t_0, t_i + \delta) = 1 - B(t_0, t_m + \delta)$$

$$B(t_0, t_m + \delta) = \frac{1 - \delta S(t_0, t_m, \delta) \sum_{i=0}^{n-1} B(t_0, t_i + \delta)}{S(t_0, t_m, \delta)\delta + 1}$$

We can use this iterative equation to create the entire LIBOR curve from seven days to thirty years. A cubic spline (which features continuous derivatives across the spline while matching the observed data) is fit to the bootstrapped curve.