On the Complexity of Winner Determination and Strategic Control in Conditional Approval Voting^{*}

Evangelos Markakis

Georgios Papasotiropoulos

Athens University of Economics and Business

MARKAKIS@AUEB.GR

GPAPASOTIROPOULOS@AUEB.GR

Abstract

We focus on a generalization of the classic Minisum approval voting rule, introduced by Barrot and Lang (2016), and referred to as Conditional Minisum (CMS), for multi-issue elections with preferential dependencies. Under this rule, voters are allowed to declare dependencies between different issues, but the price we have to pay for this higher level of expressiveness is that we end up with a computationally hard rule. Motivated by this, we first focus on finding special cases that admit efficient algorithms for CMS. Our main result in this direction is that we identify the condition of bounded treewidth (of an appropriate graph, emerging from the provided ballots) as the necessary and sufficient condition for exact polynomial algorithms, under common complexity assumptions. We then move to the design of approximation algorithms. For the (still hard) case of binary issues, we identify natural restrictions on the voters' ballots, under which we provide the first multiplicative approximation algorithms for the problem. The restrictions involve upper bounds on the number of dependencies an issue can have on the others and on the number of alternatives per issue that a voter can approve. Finally, we also investigate the complexity of problems related to the strategic control of conditional approval elections by adding or deleting either voters or alternatives and we show that in most variants of these problems, CMS is computationally resistant against control. Overall, we conclude that CMS can be viewed as a solution that achieves a satisfactory tradeoff between expressiveness and computational efficiency, when we have a limited number of dependencies among issues, while at the same time exhibiting sufficient resistance to control.

Keywords: Computational Social Choice; Multi-issue Elections; Algorithms and Computational Complexity

1. Introduction

Over the years, the field of social choice theory has focused more and more on decision making over combinatorial domains (Lang & Xia, 2016), which involves settings like *multiwinner elections*, e.g. for the formation of a committee, and elections for a set of issues that need to be decided upon simultaneously, often referred to as *multiple referenda*. In this work, we focus on approval voting as a means for collective decision making on multiple issues with multiple alternatives each. Approval voting offers a simple and easy to use format for running such elections, by having voters express an approval or disapproval separately for the alternatives of each issue. There is already a range of voting rules that are based on approval ballots, including the classic Minisum solution, which for each issue selects the alternative with the highest support from the electorate, along with more recently introduced methods as outlined in the "Related Work" section, below.

^{*.} This work has evolved as an extended merge of two preliminary conference publications: (Markakis & Papasotiropoulos, 2020, 2021).

However, the rules most commonly studied for approval voting are applicable only when the issues under consideration are independent. As soon as the voters exhibit preferential dependencies between the issues, we have more challenges to handle. More precisely, voters' preferences on a specific issue may be conditioned upon the outcome of other issue(s) and this is not uncommon in practical scenarios: A resident of a municipality may wish to support public project A, only if public project B is also implemented (which she evaluates as more important); a faculty member may want to vote in favor of hiring a new colleague only if the other new hires have a different research expertise; a group of friends may want to go to a certain movie theater only if they decide to have dinner at a nearby location; festival organizers could choose to approve the inclusion of several musical acts in their lineup but decide to limit the number of acts to a small fixed number, e.g., due to budget constraints; a grant committee may approve funding for Project X, but only if Project Y didn't receive sufficient support from the committee members to be implemented. We can also consider another example with conditional preferences, taken from recommendation systems for online advertising: suppose an ad management service needs to make a personalized selection of ads, to be shown on Alice's favorite news website. For each slot (or area) in the advertising region of the site, there is a set of possible adds to choose from and the overall goal is to maximize the likelihood that Alice will click on one of these ads. Her likelihood to click depends on whether she encounters add that strongly align with her interests. If we think of the slots as corresponding to issues, a recommendation could be made by looking at the data from users "similar" to Alice (voters), and their clicking behavior (approvals). Notably, these voters have conditional preferences, as their probability of clicking on an ad is influenced when a related ad appears in a nearby slot. For instance, the probability may increase for products frequently bought together or decrease when a product is defamed by the ad of another

It is rather obvious that voting separately for each issue cannot provide a good solution in any of the above settings. Consequently, as detailed in the "Related Work" section, several approaches have been suggested to take into account preferential dependencies. Nevertheless, the majority of these works are suitable for rules where voters are required to express a ranking over the set of issues or have a numerical representation of their preferences instead of approval-based preferences. The first work that introduced a framework for expressing dependencies exclusively in the context of approval voting was by Barrot and Lang (2016). They defined the notion of a conditional approval ballot (where the voters can specify a dependency graph for the issues of the election in conjunction with their ballots) and introduced new voting rules, that generalized some of the known rules from the literature of the standard approval setting. Among the properties that were studied, it was also exhibited that, in general, a higher level of expressiveness implies higher computational complexity. More precisely, the Minisum solution (also frequently referred to as the Approval Voting rule) is known to be efficiently computable in the standard (unconditional) approval setting, but its generalization, referred to as *Conditional Minisum* (or CMS in short), was shown to be NP-hard. In the unconditional approval setting, the Minisum solution stands as the most straightforward method for selecting winning alternatives, and it has established itself as one of the primary election systems extensively examined in economic theory, political science and computational social choice, being also widely used in practice (Endriss, 2013).

Given how central the Minisum solution is, and how practical conditional ballots can be in real-life scenarios, and in light of the computational challenges presented in (Barrot & Lang, 2016), it becomes natural to investigate whether CMS admits exact algorithms for certain families of instances or approximation algorithms with provable guarantees. Progress on this front would allow us to draw conclusions on the applicability of approvalbased elections in which voters are endowed with a significant degree of expressiveness.

In addition to our paper's focus on winner determination under CMS rule, we also take a step towards examining potential threats, that could arise due to malicious behaviour within elections over interdependent issues. These explorations can, in principle, contribute to enhancing the fairness and transparency of elections, ensuring the overall integrity of the procedure, and paving the way for the development of algorithms aimed at detecting and preventing malicious attempts. In the realm of strategic considerations, the primary focal points within the computational social choice literature revolve around questions of strategyproofness and election control. Strategyproofness is the axiom that is met when no voter can increase her satisfaction with respect to the rule's outcome by misreporting her true preferences; in contrast to the unconditional case, CMS is known to be manipulable (Barrot & Lang, 2016). In our work, we focus on elections' control and we study a spectrum of scenarios where the election conductor seeks to control the election outcome, so as to align with their own preferences, through various strategic actions. While, in many instances, a controller may be able to influence the input of the election so as to successfully enforce her will, we examine, from a computational complexity perspective, the question of whether the conductor can always and in polynomial time exert control over the outcome. To be more precise, we narrow our focus to worst-case scenarios, examining the existence of instances where the conductor encounters inherent computational challenges in achieving their desired outcomes. Similar questions form a very prominent research agenda within computational social choice as it pertains to understanding the susceptibility of election systems.

Overall, our work indicates that CMS stands as a strong candidate for real-life applications as it strikes a favorable balance between voters' expressiveness and computational efficiency, all while exhibiting sufficient robustness against certain malicious efforts.

Contribution. We undertake a study of the Conditional Minisum voting rule, a.k.a. CMS, which attempts to minimize the total dissatisfaction score across all voters in conditional approval elections, from the viewpoint of algorithms and complexity. Our goals are:

- (a.) to enhance the understanding on the complexity implications due to conditional voting for the winner determination problem under a rule that is known to be efficiently computable in the absence of dependencies between issues, with respect to both exact and approximate solutions,
- (b.) to identify whether the rule is immune to, or, at least, computationally resistant against malicious control by strategic actions.

In Section 3.1, we focus on conditions that lead to exact polynomial time algorithms for computing the optimal solution under CMS voting rule. For this, we consider the intuitively simple (but still NP-hard) case, where each issue can depend on at most one other issue for every voter, and our main insight is that one can draw conclusions by looking at (undirected variant of) the global dependency graph of an instance, which is formed by taking the union of dependencies by all voters. We later generalize this result for dependencies on any constant number of issues. Restrictions on the structure of the global dependency graph allow us to identify the condition of bounded treewidth as the only restriction that leads to optimal efficient algorithms. More precisely, our results provide characterizations for the families of CMS instances that can be placed in P and FPT, implying that the condition of bounded treewidth serves as the lynchpin between expressiveness of voters' ballots and efficiency of solving the winner determination problem. These results also establish a connection with well studied classes of Constraint Satisfaction Problems, which can be of independent interest.

In Section 3.2, we provide the first multiplicative approximation algorithms for conditional approval elections of issues with binary domain, under the condition that for every voter, each issue can depend on at most one other issue. The considered family of instances, includes the set of instances that were proven to be NP-hard in (Barrot & Lang, 2016). In the corresponding graph-theoretic representation of the problem, which will be introduced in Section 2, the condition corresponds to voters with dependency graph of maximum indegree no more than 1. The main positive result of the section is an algorithm that achieves an approximation factor of 1.1037. Interestingly, our algorithm is based on a reduction to MIN SAT, an optimization version of SAT that has rarely been applied in computational social choice (in contrast to MAX SAT). The result is contingent upon an additional, but well-motivated from the perspective of social choice, assumption regarding the number of approved alternatives per issue in a voter's ballot. Imposing such a further requirement might at first seem demanding, however, we have also established a strong negative result: in the absence of further assumptions, no algorithm can attain any bounded multiplicative approximation guarantee, even for instances with binary domains and even if for every voter, each issue depends on at most one other issue. Concluding the section, we put forth some additional (and similar in flavor) assumptions that, when satisfied, also enable the existence of provable approximation guarantees, albeit non-constant. Interestingly, these results also relate the considered voting rule with classical algorithmic problems.

Moving on, in Section 4, we initiate the algorithmic study of some standard notions of election control for CMS. The problems we examine concern the attempt by an external agent to enforce the election of certain alternative(s) in either one or every issue under consideration, by adding or deleting either voters or alternatives. We consider a total of 8 variants of this question, depending on the number of issues to be controlled and on whether we have addition or deletion of voters/alternatives. We provide a set of 18 computational complexity results that give a complete picture with respect to the crucial parameters of the input in every one of the considered problems. Our findings reveal that CMS is sufficiently computationally resistant, against such moves.

Related Work. Approval voting for multi-issue elections has gained great attention in the recent years, driven by its simplicity and practical potential. Apart from the classic Minisum solution (Weber, 1978; Brams & Fishburn, 1978, 1982; Laslier & Sanver, 2010), other rules have also been considered, such as the Minimax solution (Brams, Kilgour, & Sanver, 2007), Satisfaction Approval Voting (Brams & Kilgour, 2015), families of rules based on Weighted Averaging Aggregation (Amanatidis, Barrot, Lang, Markakis, & Ries, 2015), Proportional Approval Voting and Chamberlin-Courant. The last two rules, as

well as Minisum, can be captured by the general family of Thiele voting rules; for these (and other approval based rules) we refer to the very recently published book (Lackner & Skowron, 2023) and to the surveys (Brams & Fishburn, 2010; Kilgour, 2010). None of these rules however allow voters to express dependencies. The first work that exclusively took this direction for approval-based elections is by Barrot and Lang (2016). Namely, three voting rules were proposed for incorporating such dependencies (including the Conditional Minisum rule that we consider here) and some of their properties were studied, mainly on the satisfiability of certain axioms. Conditional approval ballots have a clear resemblance with the well-studied model of CP-nets (Boutilier, Brafman, Domshlak, Hoos, & Poole, 2004), which is a graphical representation of voters' preferences depicting conditional dependence and independence of preference statements under a ceteris paribus (all else being equal) interpretation, but, as it has been highlighted in (Barrot & Lang, 2016) the two frameworks define different semantics and are incomparable.

Even if one moves away from approval-based elections, the presence of preferential dependencies remains a major challenge when voting over combinatorial domains. Several methodologies have been considered achieving various levels of trade-offs between expressiveness and efficient computation. Some representative examples include, among others, sequential voting (Lang & Xia, 2009; Airiau, Endriss, Grandi, Porello, & Uckelman, 2011; Dalla Pozza, Pini, Rossi, & Venable, 2011; Xia & Conitzer, 2012), or completion principles for partial preferences (Laffond & Lainé, 2009; Cuhadaroglu & Lainé, 2012). Analogous attempts to increase the expressiveness of agents' ballots have been also examined in other subfields of computational social choice; indicatively we refer to Participatory Budgeting (Rey, Endriss, & de Haan, 2023; Jain, Sornat, & Talmon, 2020), Judgement Aggregation (Grandi & Endriss, 2010) and Liquid Democracy (Colley, Grandi, & Novaro, 2021).

Finally, in our work we also consider versions of election control that fall within the standard approaches that have been used for studying the complexity of affecting election outcomes. For an extensive study on this topic, we refer to (Faliszewski & Rothe, 2016). Indicatively, the study of such problems with adding or deleting voters or alternatives began with (Bartholdi III, Tovey, & Trick, 1992) and some subsequent works are (Gupta, Roy, Saurabh, & Zehavi, 2022; Yang, 2019; Hemaspaandra, Hemaspaandra, & Rothe, 2007; Faliszewski, Hemaspaandra, & Hemaspaandra, 2011; Liu, Feng, Zhu, & Luan, 2009; Meir, Procaccia, Rosenschein, & Zohar, 2008).

2. Formal Background

Let $I = \{I_1, \ldots, I_m\}$ be a set of *m* issues, where each issue I_j is associated with a finite domain D_j of alternatives. An *outcome* is an assignment of a value for every issue, and let $D = D_1 \times D_2 \times \cdots \times D_m$ be the set of all possible outcomes. Let also $V = \{1, \ldots, n\}$ be a group of *n* voters who have to decide on a common outcome from *D*.

Voting Format. To express dependencies among issues, we mostly follow the format described in (Barrot & Lang, 2016). Each voter $i \in [n]$ is associated with a directed graph $G_i = (I, E_i)$, called *dependency graph*, whose vertex set coincides with the set of issues. A directed edge (I_k, I_j) means that issue I_j is affected by I_k . We also let $N_i(I_j)$ be the (possibly empty) set of direct predecessors of issue I_j in G_i . We first explain briefly how the voters are expected to submit their preferences, before giving the formal definition.

For an issue I_j that has no predecessors in G_i (in other words, its in-degree is 0), voter i is allowed to cast an unconditional approval ballot, stating the alternatives of D_j that are approved by her. In the case that issue I_j has a positive in-degree in G_i , then let $\{I_{j_1}, I_{j_2}, \ldots, I_{j_k}\} \subseteq I$ be all its direct predecessors (also called in-neighbors). Voter i then needs to specify all the combinations that she approves in the form $\{t:r\}$ where $r \in D_j$, and $t \in D_{j_1} \times D_{j_2} \times \cdots \times D_{j_k}$. Every such combination $\{t:r\}$ signifies the satisfaction of voter i with respect to issue I_j in a given outcome, when that outcome contains all alternatives in t as well as the alternative r for the issue I_j . Both cases of zero and positive in-degree for an issue can be unified in the following definition of conditional approval ballots.

Definition 1. A conditional approval ballot of a voter *i* over issues $I = \{I_1, \ldots, I_m\}$ with domains D_1, \ldots, D_m respectively, is a pair $B_i = \langle G_i, \{A_j, j \in [m]\} \rangle$, where G_i is the dependency graph of voter *i*, and for each issue I_j , A_j is a set of conditional approval statements in the form $\{t:r\}$ with $t \in \prod_{k \in N_i^-(I_i)} D_k$, and $r \in D_j$.

To simplify the presentation, when a voter has expressed a common dependency for k > 1 alternatives of an issue I_j , we can group them together and write $\{t : \{d_j^1, d_j^2, \ldots, d_j^k\}\}$, instead of $\{t : d_j^1\}$, $\{t : d_j^2\}$, ..., $\{t : d_j^k\}$. Additionally, for every issue I_j with in-degree 0 by some voter i, a vote in favor of d_j will be written simply as $\{d_j\}$, instead of $\{\emptyset : d_j\}$.

An important quantity for parameterizing families of instances is the maximum indegree¹ of each graph G_i , namely $\Delta_i = \max_{j \in [m]} \{|N_i^-(I_j)|\}$. Let also $\Delta = \max_{i \in [n]} \Delta_i$. Given a voter *i* with conditional ballot B_i , we will denote by B_i^j the restriction of her ballot to issue I_j . Moreover, a conditional approval voting profile is given by a tuple P = (I, D, V, B), where $B = (B_1, B_2, \ldots, B_n)$.

Definition 2. The global dependency graph of a set of voters is the undirected simple graph that emerges from ignoring the orientation of edges in the graph $(I, \bigcup_{i \in [n]} E_i)$, where E_i is the edge set of the dependency graph of voter *i*.

Example 1. As an illustration, we consider 3 co-authors of some joint research who have to decide on 3 issues: whether they will *work* more before the submission deadline on obtaining new theorems, whether they have enough material to split their work into two, or even *multiple*, papers or submit all their results in a single submission, and whether they should invite a new *co-author* to work with them because of his insights that can help on improving their results. The first author insists on more work before the submission, additionally he approves the choice of two submissions if and only if they work more on new theorems. Furthermore, he does not want to have a new co-author if and only if they split their work. The second author does not have time for more work before the deadline, he has no strong opinion on multiple submissions, approving both alternatives, and he agrees with inviting a new co-author only if they decide both to work more for new results and to submit a single paper. Finally, the last author is interested in working more and in splitting their work and she does not have a strong opinion on whether she prefers to invite a new co-author or not, unless they all decide not to work more neither to make more than a single submission, in which case she disagrees with such an invitation.

^{1.} When Δ_i is large for some voter *i*, the input might become exponentially large. Alternatively, one could try a succinct way of representing ballots using propositional formulae. We will not examine further this issue, since for the cases that we consider, the in-degree is constant.

More formally, let $I = \{I_1, I_2, I_3\}$ be the aforementioned issues where $D_1 = \{w, \overline{w}\}$, $D_2 = \{m, \overline{m}\}, D_3 = \{c, \overline{c}\}$. The dependency graphs and the voters' preferences follow.

$=I_2$	voter 1	voter 2	voter 3
$I_1 \sim \downarrow I_2$	w	$\{\overline{w}, m, \overline{m}\}$	$\{w,m\}$
	$\overline{w}:\overline{m}$	$wm:\overline{c}$	$wm:\{c,\overline{c}\}$
I_2 I_3	w:m	$\overline{w}m:\overline{c}$	$\overline{w}m:\{c,\overline{c}\}$
	$m:\overline{c}$	$w\overline{m}:c$	$w\overline{m}:\{c,\overline{c}\}$
	$\overline{m}:c$	$\overline{wm}:\overline{c}$	$\overline{wm}:\overline{c}$

Figure 1: The dependency graph of voter 1 (up Figure 2: The conditional ballots of the voters. left), voters 2 and 3 (down left) and the global dependency graph (right).

Voting Rule. In this work, we study a generalization of the classic Minisum solution in the context of conditional approval voting. To do so, we firstly define a measure for the dissatisfaction of a voter given an assignment of values to all the issues, using the following generalization of Hamming distance.

Definition 3. Given an outcome $s = (s_1, s_2, \ldots, s_m) \in D$, we say that voter *i* is dissatisfied (or disagrees) with issue I_j , if for the projection of *s* on $N_i(I_j)$, say *t*, it holds that $\{t : s_j\} \notin B_j^j$. We denote as $\delta_i(s)$ the total number of issues that dissatisfy voter *i*.

Coming back to Example 1, the values of $\delta_i(s)$ for every outcome s and voter i follow.

$\delta_i(\cdot)$	wmc	$wm\overline{c}$	$w\overline{m}c$	$w\overline{mc}$	$\overline{w}mc$	$\overline{w}m\overline{c}$	$\overline{wm}c$	\overline{wmc}
voter 1	1	0	1	2	3	2	1	2
voter 2	2	1	1	2	1	0	1	0
voter 3	0	0	1	1	1	1	3	2

The rule that our work deals with is *Conditional Minisum* (CMS) and outputs the outcome that minimizes the total number of disagreements over all voters. To simplify notation, we will use CMS to refer both to the voting rule and to the related algorithmic problem; the exact meaning will always be clear from the context. Formally, the algorithmic problem that we study is as follows.

	CONDITIONAL MINISUM (CMS)
Given:	A voting profile P with m issues and n voters casting conditional approval ballots.
Output:	A boolean assignment $s^* = (s_1^*, \ldots, s_m^*)$ to all issues that achieves $\min_{s \in D} \sum_{i \in [n]} \delta_i(s)$.

For Example 1, we can see that the Conditional Minisum solution would prescribe to the authors to work more for new results, to split their work into two submissions, and not to invite a new co-author, which corresponds to the outcome $\{wm\bar{c}\}$.

If the global dependency graph of an instance is empty, i.e., $\Delta_i = 0$ for every voter *i*, then the election degenerates to Unconditional Minisum which is simply the classic Minisum rule in approval voting over multiple independent issues.

Finally, in the sequel, we will extensively make use of the *treewidth of a graph G*, denoted as tw(G). For the relevant definition, we refer to (Robertson & Seymour, 1986) or to any textbook of parameterized complexity such as (Cygan, Fomin, Kowalik, Lokshtanov, Marx, Pilipczuk, Pilipczuk, & Saurabh, 2015).

3. Winner Determination

In this section we focus on the winner determination problem, under the Conditional Minisum voting rule. In Section 3.1, we mainly present a characterization (subject to certain complexity theory assumptions) of the instances which admit polynomial time algorithms for computing the optimal outcome of the rule, whereas in Section 3.2, our focus is on approximate solutions, for computationally more demanding instances.

3.1 Optimal Algorithms

The price we pay for the higher expressiveness of CMS, compared to the classical Minisum solution, is its increased complexity. Here, we focus on understanding the properties that allow CMS to be implemented in polynomial time. For this, we firstly stick to the case where $\Delta_i \leq 1$ for every voter i, which is already NP-hard, and at the same time forms the most obvious, first-step generalization of Unconditional Minisum to the setting of dependencies. Then, in Section 3.1.1, we generalize our results for profiles of bounded Δ_i , for every voter *i*. To investigate what further restrictions can make the problem tractable, we utilize the global dependency graph of an instance, defined in Section 2, as the aggregation of all the dependencies of the voters into a single graph. To see how to exploit the global dependency graph, it is instructive to inspect the NP-hardness proof for CMS in (Barrot & Lang, 2016), which holds for instances where $\Delta_i = 1$ for every voter *i*, and each dependency graph is acyclic. Examining the profiles created in that reduction, we notice that no restrictions can be stated for the form of the global dependency graph corresponding to the produced instances. This holds since, an acyclic dependency graph for every voter does not necessarily lead to an acyclic global dependency graph and furthermore, the bounded in-degree in each G_i , does not imply a constant upper bound for the maximum in-degree of the global graph.

Our insight is that it may not be only the structure of each voter's dependency graph that causes the problem's hardness, but in addition, the absence of any structural property on the global dependency graph. Motivated by this, we investigate conditions for the global dependency graph, that enable us to obtain the optimal solution in polynomial time. Our findings reveal that this is indeed feasible for the classes of graphs with constant treewidth.

In our results, we make extensive use of *Constraint Satisfaction Problems* (CSPs). A CSP instance is described by a tuple (V, D, C), where V is the set of variables, D is the Cartesian product of the domains of the variables, and C is a set of constraints. Each constraint involves a subset of the variables, and is represented by all the combinations of variables that make it satisfied. We will pay particular attention to the so-called *binary* CSPs, where each constraint involves at most two variables. The decision problem for a CSP asks whether we can find an assignment to the variables of V so that all constraints of C are satisfied, whereas a natural optimization version (Freuder & Wallace, 1992) is to minimize the number of unsatisfied constraints. When analyzing CSPs, a useful concept in the literature is the *primal* or *Gaifman* graph of an instance, defined below.

Definition 4. The primal (or Gaifman) graph of a CSP instance is an undirected graph, whose vertices are the variables of the instance and there is an edge between two vertices, if and only if they co-appear in at least one constraint.

The proof of the following theorem is based on formulating our problem as minimizing the number of unsatisfied constraints in an appropriate binary CSP instance, whose primal graph has constant treewidth. For these classes of CSPs, one can then use known results from (Freuder, 1990) or (Koster, van Hoesel, & Kolen, 2002) for solving them efficiently.

Theorem 1. If the global dependency graph of a CMS instance with $\Delta_i \leq 1$, for every voter *i*, has constant treewidth, then CMS can be implemented in polynomial time, even with an arbitrary domain cardinality for each issue.

Proof. Consider an instance P = (I, D, V, B) of CMS with n voters and m issues, and let G be its global dependency graph. Suppose the treewidth of G is bounded by $k \in \mathcal{O}(1)$, and let d be the maximum cardinality among the domains. We form an instance of the minimization version of binary CSP, with $m \cdot n$ constraints, where each constraint expresses the satisfaction of a specific voter for a specific issue.

Recall that we have assumed the maximum in-degree of every voter's dependency graph is at most one, thus each constraint in the CSP instance that we construct involves at most two variables, which means that the obtained CSP is indeed binary. Also, we can express each constraint by providing at most d^2 combinations of the two involved variables (i.e., the combinations that satisfy the constraint). Hence, the construction of the CSP instance can be done in polynomial time.

Since each constraint that involves two variables² corresponds to an edge of the global dependency graph and constraints with exactly one variable do not contribute any edges neither to the primal nor to the global dependency graph, the following can be easily verified.

Claim 1. The primal graph of the produced CSP instance is identical to the global dependency graph of the CMS instance.

Therefore, CMS has been formulated as minimizing the number of unsatisfied constraints in a binary CSP with primal graph of constant treewidth and these classes of CSPs are solvable in $\mathcal{O}(n^k)$ time by (Freuder, 1990)³ or (Koster et al., 2002).

We additionally highlight that the above theorem can be generalized when there is a weight w_i for each voter i so that the objective becomes the weighted sum of the dissatisfaction scores.

In trying to move away from treewidth-based assumptions, a natural question is whether we can solve other classes of instances, containing graphs of non-constant treewidth, by focusing on other parameters of the problem. Quite surprisingly, it turns out that bounded treewidth is essentially the only property that can yield efficiency guarantees. To establish this claim, we will first show a "reverse" direction to Theorem 1, namely that binary CSPs

^{2.} For uniformity, we could add dummy issues in the CMS instance (resp. dummy variables in the CSP instance) so that the final CSP only has constraints with exactly two variables.

^{3.} In fact, the original results in (Freuder, 1990) do not deal with the optimization version, but as demonstrated in later works (see e.g., Proposition 4.3 from (Knop, Koutecký, Masarík, & Toufar, 2019), it can be extended for this version as well.

can be reduced to solving CMS. Hence, together with Theorem 1, this means that CMS is computationally equivalent to binary CSPs, and thus to any other problem for which the same result has been already established, e.g., the PARTITIONED SUBGRAPH ISOMORPHISM problem (Marx, 2010).

Theorem 2. Every binary CSP with primal graph G, can be reduced in polynomial time to a CMS instance with $\Delta_i \leq 1$ for every voter i, and with G as the global dependency graph.

Proof. For convenience, we will work with the standard decision version of CSP where one asks if there is a solution that satisfies all the constraints.

Let P be a binary CSP instance, and without loss of generality, assume that every constraint involves exactly two variables (which can be enforced by the addition of dummy variables). We construct a CMS instance P', where the issues correspond to the variables and the voters correspond to the constraints of P. In particular, for every variable x_j of the CSP instance, we add an issue I_j and for every constraint we add a voter, with the following preferences: let x_j, x_k be the two variables involved in the constraint. We pick one of the two variables (arbitrarily), say x_k , and we set I_k as the issue that the voter cares about, conditioned on I_j . We also set her conditional ballot for issue I_k in such a way, so that the voter becomes satisfied precisely for all combinations of values for x_j and x_k that make the constraint satisfied. The voter is also satisfied unconditionally with every outcome for every other issue of the produced instance. Obviously, the dependency graph of every voter has maximum in-degree equal to one.

As an example, suppose that a constraint is of the form $x_1 \vee x_2$ and the variables x_1, x_2 have binary domain. Then we introduce a new voter, and two issues, I_1 , I_2 (the issues may have been introduced already by other constraints in the instance), and we can select I_2 as being dependent on I_1 . The conditional ballot regarding the satisfaction of the voter for I_2 is $\{x_1 : x_2\}, \{\overline{x_1} : x_2\}, \{x_1 : \overline{x_2}\}$. In addition, the voter has an unconditional ballot for I_1 , in the form $\{x_1, \overline{x_1}\}$, thus approving every value for I_1 .

To complete the reduction, we consider the decision version of CMS where we ask if there is an assignment with no dissatisfactions, i.e., the instance P' has an affirmative solution only when all voters are satisfied with all the issues. It is obvious that this is a polynomial time reduction (the conditional ballot of each voter for her single issue of interest can be described in $\mathcal{O}(d^2)$ time, where d is the maximum domain cardinality of the CSP variables). It is quite obvious also that every edge from the primal graph of P corresponds to an edge in the global dependency graph of P', and vice versa. Hence:

Claim 2. The primal graph of CSP instance P is identical to the global dependency graph of the CMS instance P'.

Finally, it remains to see that there exists a solution to P' if and only if there exists a solution to P. Indeed, any solution to P' corresponds to an assignment of values to the issues such that all voters are satisfied with all issues, which means that all the constraints of the CSP instance P are satisfied. The converse is also easily verified.

Theorem 2 allows us to apply some well known hardness results on binary CSPs, namely (Grohe, 2007; Grohe, Schwentick, & Segoufin, 2001), which imply that one cannot hope to have an efficient algorithm for a class of CMS instances, if the class contains instances

with non-constant treewidth. Hence, Theorem 1 is essentially tight, and this resolves the problem of finding a characterization for polynomial time solvability of CMS, subject to a standard complexity theory assumption. This is summarized in the following corollary.

Corollary 1 (implied by Theorems 1 and 2 and by (Grohe, 2007)). Let \mathcal{G} be a recursively enumerable (e.g., decidable) class of graphs, and let $CMS(\mathcal{G})$ be the class of instances with a global dependency graph that belongs to \mathcal{G} , and with $\Delta_i \leq 1$ for every voter *i*. Assuming $FPT \neq W[1]$, there is a polynomial algorithm for $CMS(\mathcal{G})$ if and only if every graph in \mathcal{G} has constant treewidth.

Proof. If \mathcal{G} is a class of graphs, as in the statement, then by Theorem 2, an algorithm for the class of CMS instances whose global dependency graph belongs to \mathcal{G} implies an algorithm for the CSP instances whose primal graph belongs to \mathcal{G} . The proof can now be completed by applying the hardness results for binary CSPs by (Grohe, 2007; Grohe et al., 2001). \Box

Remark 1. If we strengthen the complexity assumption used in Corollary 1, from FPT \neq W[1] to the Exponential Time Hypothesis (ETH), we can obtain an even stronger impossibility. In particular, by exploiting the result of (Marx, 2010), and the proof of Theorem 2, we can show that under ETH, one cannot even hope for an algorithm on CMS(\mathcal{G}) that runs in time $f(G)||P||^{o(tw(G))/log(tw(G))}$, where ||P|| is the size of the CMS instance and $G \in \mathcal{G}$. This implies that the running time $\mathcal{O}(n^{tw(G)})$ of the algorithm from Theorem 1 is the best possible up to an $\mathcal{O}(\log(tw(G)))$ factor in the exponent.

3.1.1 Generalizations to higher in-degrees

We highlight that Theorem 1 cannot be immediately generalized so as to apply to instances where $\Delta_i \geq 2$ for some voter *i*, since in that case the global dependency graph will not necessarily coincide with the primal graph of the corresponding CSP that we constructed in the proof of Theorem 1 (which is an essential part of the proof). In order to obtain a result for higher in-degrees, we introduce the following definition, which is a generalization of the global dependency graph, and where we simply replace a vertex and its in-neighbors by a clique on the same set of vertices.

Definition 5. The extended global dependency graph of a set of voters is the undirected (simple) graph $(I, \bigcup_{i \in [n]} \tilde{E}_i)$, where \tilde{E}_i is the edge set of a graph that corresponds to voter i and is created by enforcing an undirected clique for every issue I_j and any voter i, on the set $N_i(I_j) \cup \{I_j\}$.

Note that for the cases that $\Delta \leq 1$, the extended global dependency graph of an instance coincides with the global dependency graph (and hence with the primal graph created in the proof of Theorem 1). The crucial observation now is that as long as $\Delta \in O(1)$, an instance of CSP equivalent to the initial CMS instance, can be created in polynomial time by following very closely the proof of Theorem 1. And most importantly, even though the extended global dependency graph of the CMS instance does not coincide with the global dependency graph, it does coincide with the primal graph of the created CSP instance, which is all we need. We stress also that one of the reasons we need $\Delta \in O(1)$, is to ensure that we can specify all the combinations that satisfy a constraint in polynomial time (namely by specifying the at most d^{Δ} satisfying combinations). To finalize the argument for the generalization, note that the created CSP instance will no longer be a binary CSP (i.e., it will not have at most two variables in each constraint). Nevertheless, these instances will have at most a constant number of variables in each constraint, due to Δ being constant, and they are still tractable as long as the primal graph of the CSP has bounded treewidth (Freuder, 1990). Hence, our discussion can be summarized by the following theorem, which is indeed a generalization of Theorem 1 for instances of higher in-degrees.

Theorem 3. If the extended global dependency graph of a CMS instance with $\Delta_i \in O(1)$ for every voter *i*, has constant treewidth, then CMS can be implemented in polynomial time, even with an arbitrary domain cardinality for each issue.

Finally, we can also obtain a generalization of Theorem 2 (the exact same arguments apply with the global dependency graph being replaced by the extended global dependency graph). This leads to the following characterization regarding instances with higher indegrees, which is the analog of Corollary 1.

Corollary 2. Let \mathcal{G} be a recursively enumerable class of graphs, and let $CMS(\mathcal{G})$ be the class of instances with an extended global dependency graph that belongs to \mathcal{G} , and with $\Delta_i \in O(1)$ for every voter *i*. Assuming $FPT \neq W[1]$, there is a polynomial algorithm for $CMS(\mathcal{G})$ if and only if every graph in \mathcal{G} has constant treewidth.

We finally note that Remark 1 also applies here, but again for the treewidth of the extended global dependency graph.

3.1.2 PARAMETERIZED COMPLEXITY OF CMS

The algorithm used in the proof of Theorem 1, runs in time exponential in tw(G), where G is the global dependency graph and thus it places CMS in XP w.r.t the treewidth parameter. One can wonder if anything more can be said concerning the fixed parameter tractability of the problem. Given the equivalence of our problem for $\Delta_i \leq 1$, for every voter *i*, with binary CSP, we can use existing results (Samer & Szeider, 2010; Gottlob & Szeider, 2008) to extract some further characterizations and obtain an almost complete picture with respect to the most relevant parameters. On the positive side, we can see that our problem is in FPT with respect to the parameter "treewidth + domain size". On the negative side, we cannot hope to prove FPT only w.r.t the one of the two parameters, independent of the other, as stated below.

Corollary 3. When $\Delta_i \in \mathcal{O}(1)$ for every voter *i*, CMS is in FPT w.r.t the parameter tw+d, where tw is the treewidth of the extended global dependency graph and *d* is the maximum domain size. Moreover, even when $\Delta_i \leq 1$ for every voter *i*, it is W[1]-hard with respect to tw and with respect to *d*.

Proof. First, let us introduce some notation for ease of presentation. Given a set of parameters S, we denote as $\Pi\{S\}$ the parameterized version of a problem Π , having all variables in S as parameters. $\Pi\{S\}$ is in FPT if every instance I of Π can be solved in time $\mathcal{O}(f(S)|I|^c)$ for some constant c, and a computable function f, independent of any variable of Π other than the parameters in S. For a CSP instance we will denote by tw' the treewidth of its primal graph, by d' the maximum domain size of every variable, and by *arity* the maximum number of variables that co-appear in a constraint.

To prove the positive statement, we exploit the fact that $\text{CSP}\{arity, d', tw'\}$ is in FPT by (Grohe et al., 2001). This trivially implies that for CSP instances of constant arity, we have that $\text{CSP}\{d', tw'\}$ is in FPT. We can now use our Theorem 3. In particular, if we have a CMS instance, where $\Delta_i \in \mathcal{O}(1)$, and where d is the maximum domain size and tw is the treewidth of the extended global dependency graph, Theorem 3 shows that we can reduce this to solving a CSP instance of constant arity and with d' = d and tw' = tw. Hence, we have that $\text{CMS}\{tw, d\}$ is in FPT, when $\Delta_i \in \mathcal{O}(1)$.

To prove the negative statements, we use the following definition: A set of parameters S dominates a set S' if whenever all parameters of S' are bounded by some constants, all parameters of S are bounded too. In (Samer & Szeider, 2010) (Theorem 1 therein), it was proved that $CSP_{bin}\{arity, tw'\}$ and $CSP_{bin}\{arity, d'\}$ are W[1]-hard, where CSP_{bin} denotes the class of binary CSP instances. It is trivial to see that the set $S = \{tw'\}$ dominates the set $S' = \{arity, tw'\}$. Hence, by utilizing Lemma 1 in (Samer & Szeider, 2010), we obtain that $CSP_{bin}\{tw'\}$ is W[1]-hard and the same is true also for $CSP_{bin}\{d'\}$. Given the reduction established in our Theorem 2, of binary CSPs with parameters tw' and d' to CMS instances with $\Delta_i \leq 1$ and with tw = tw' and d = d', we can conclude that both $CMS\{tw\}$ and $CMS\{d\}$ are W[1]-hard too.

We conclude the subsection by noting that the instances captured by the assumptions we have made are indeed meaningful in multi-issue elections with logically dependent issues. Firstly, we mostly considered instances where $\Delta_i \leq 1$ for every voter *i*, which is nontrivial (NP-hard to solve the winner determination problem), but still the obvious first-step generalization of the traditional (minisum) approval voting rule. Secondly, the main positive result was for the case where the global dependency graph has a bounded treewidth. This can allow e.g., for paths, where we could think of the issues as being ordered on a line, with a sequential dependence between them. Likewise, when the global dependency graph forms a tree, we can again have a hierarchy regarding dependencies (e.g., a star graph can arise when there is a central issue, the decision for which influences the satisfaction of voters on the remaining issues). Going further, a constant treewidth allows for even more complex dependencies among issues, but still well-structured.

3.2 Approximation Algorithms

It is well known that a Minisum solution can be efficiently computed when there are no dependencies (Brams et al., 2007). In contrast to this, CMS is NP-hard even when all the issues have a binary domain and there is only a single dependence per voter, i.e., when every voter's dependency graph has just a single edge (Barrot & Lang, 2016). Given this hardness result, it is natural to resort to the framework of approximation algorithms. The only known result from this perspective is an algorithm by (Barrot & Lang, 2016), with a differential approximation ratio of $4.34/(m \sum_{j \in I} 2^{|N^-(j)|} + 4.34)$, for the case of a common acyclic dependency graph, where $N^-(j)$ is the set of common in-neighbors of issue j (for each voter i and issue j, $N_i^-(j) = N^-(j)$). However, differential approximations (we refer to (Demange, Grisoni, & Paschos, 1998) for the definition of this concept) form a less typical approach in the field of approximation algorithms. Instead, we focus on the more

standard framework of *multiplicative* approximation algorithms, as treated also in common textbooks (Vazirani, 2003; Williamson & Shmoys, 2011). We say that an algorithm for a minimization problem achieves a multiplicative ratio of $\alpha \geq 1$, if for every instance I, it produces a solution with cost at most α times the optimal. We stress that a differential approximation ratio for minimization problems does not in general imply any multiplicative approximation ratio (Bazgan & Paschos, 2003).

We start first with a rather strong negative result in terms of the viability of approximate solutions. The main result of the previous subsection was the hardness of computing optimal outcomes, which is implied by Theorem 2. In fact, the proof of Theorem 2 implies also the following important multiplicative inapproximability about CMS.

Corollary 4. Even when $\Delta_i \leq 1$ for every voter *i*, it is NP-hard to obtain any finite approximation ratio for CMS.

Proof. If we look again at the proof of Theorem 2, we can see that we have reduced the solution of a binary CSP instance to deciding whether a CMS instance admits a solution of cost zero, i.e., a solution where all voters are satisfied. Given the hardness of binary CSPs, we conclude then that deciding if a CMS instance has optimal cost equal to zero is NP-hard. Suppose now that we could obtain an approximation algorithm with some finite approximation ratio for every instance. This immediately means that we could use this algorithm to distinguish between instances that have an optimal cost of zero (where the algorithm would have to return the optimal solution by the definition of approximation ratio) from the remaining instances (where the algorithm would return some solution with a positive cost). Hence we would have solved an NP-hard problem. \Box

Therefore, a polynomial time algorithm with a bounded multiplicative approximation guarantee, could only be possible under further assumptions. Our main contribution in this subsection is the first class of multiplicative approximation algorithms for some special cases of CMS. Sticking to the already hard class of binary domains and $\Delta_i \leq 1$, for every voter *i* (which includes the instances in which every voter has one edge, considered in the hardness result of (Barrot & Lang, 2016)), we focus on instances that satisfy an assumption motivated by the fact that in the unconditional case, allowing voters to approve at most a single alternative per issue is already an interesting and well-studied voting scenario, which corresponds to the multi-issue analog of elections under the classical plurality setting.

Definition 6. Consider a CMS instance with binary domains, and where the dependency graph of every voter *i* satisfies $\Delta_i \leq 1$. The instance is called 1-approval, if for every issue I_j that is dependent on some issue I_k according to the preferences of a voter *i*, it holds that *i* can be satisfied only with one pair, say $\{x_k : x_j\}$, w.r.t. I_j , where $x_k \in D_k$ and $x_j \in D_j$. No restrictions are imposed to the number of approved alternatives for unconditional ballots.

To obtain a positive result, we first make use of known approximation algorithms for MIN k-SAT, a minimization version of SAT, where we are given a set of m clauses in k-CNF and we search for a boolean assignment so as to minimize the total number of satisfied clauses. Interestingly, minimization versions of SAT have rarely been applied in the context of computational social choice, see e.g., (Lang, Mengin, & Xia, 2018). In fact it has hardly ever been used as a tool for obtaining approximation algorithms for other problems (we are

only aware of an application for certain string comparison problems (Goldstein, Kolman, & Zheng, 2005)). The use of MAX SAT is much more common, but for the case of CMS, and for multiplicative approximation guarantees, it does not seem convenient to exploit algorithms for maximisation problems. In a nutshell, if we use an approximation algorithm for MAX SAT, the conversion from the solution of a maximization problem to that of a minimization one that we have here, does not preserve a good approximation ratio for our objective function⁴. The main positive result of this subsection follows.

Theorem 4. Let \mathcal{F} be the family of 1-approval CMS instances, with binary domains and with $\Delta_i \leq 1$ for every voter *i*. Then any α -approximation algorithm for MIN 2-SAT yields an α -approximation algorithm for the family \mathcal{F} . In particular, we can have a polynomial time 1.1037-approximation for any CMS instance in \mathcal{F} .

Proof. We present a reduction to MIN 2-SAT that preserves the approximation factor in the case where the given CMS instance is 1-approval. We first present a general reduction for any instance with $\Delta_i \leq 1$ for every voter i, which could be of broader interest. Later on, we will see how we can exploit this construction for 1-approval instances. Therefore, consider an arbitrary instance P of n voters, with $\Delta_i \leq 1$ for every voter i, and let $I = \{I_1, \ldots, I_m\}$ be the set of issues. We first create a logical formula C_{ij} , for every voter $i \in V$, and every issue $I_j \in I$, which indicates the cases where voter i is not satisfied with the outcome on I_j . For every issue I_j , recall that $D_j = \{d_j, \overline{d_j}\}$ is its domain, and x_j will be the corresponding boolean variable in the construction of C_{ij} .

For this we consider two cases. The first and easier case is when for a voter i, and issue I_j , $N_i^{-}(I_j) = \emptyset$. All possible forms of B_i^j are depicted in the first row of Table 1, whereas the corresponding formula is shown in the second row.

$$\begin{array}{c|c} B_i^j & \emptyset & \{d_j\} & \{\overline{d_j}\} & \{d_j, \overline{d_j}\} \\ \hline C_{ij} & x_j \lor \overline{x}_j & \overline{x}_j & x_j & \emptyset \end{array}$$

Table 1: The formula when issue I_i has no predecessor in G_i .

On the other hand, if I_j has an in-neighbor (it can have only one by our assumptions), say $I_k \in I$, we set C_{ij} equal to the disjunction of all combinations of outcomes on issues I_j and I_k that dissatisfy voter *i* with respect to I_j . To illustrate this construction, we describe an example with 4 voters, 2 issues $I = \{I_k, I_j\}$ and for every voter *i*, $G_i = \{I, \{I_k, I_j\}\}$. The preferences for issue I_j are shown in Table 2. Namely, for i = 1, 2, 3, 4, the first cell in the *i*-th row depicts B_i^j from which C_{ij} can be obtained as the disjunction of the ticked expressions in the remaining of the *i*-th row.

Claim 3. For an outcome (s_1, \ldots, s_m) of the issues and the corresponding assignment to the boolean variables x_1, \ldots, x_m , voter *i* is dissatisfied with I_j if and only if the formula C_{ij} is true.

The constructed formula C_{ij} is in DNF. To continue, we will need to make a conversion to CNF, which is easy to do given its small size as per the following lemma.

^{4.} The differential approximation result of (Barrot & Lang, 2016) was based on the use of MAX SAT. But as stated earlier, this does not imply any non-trivial multiplicative approximation for CMS.

B_i^j	$(x_k \wedge x)$	$_{j})\left(x_{k}\wedge \overline{x}_{j} ight)$	$(\overline{x}_k \wedge x)$	$(\overline{x}_k \wedge \overline{x}_j)$
Ø	\checkmark	\checkmark	\checkmark	\checkmark
$\{d_k:d_j\}$		\checkmark	\checkmark	\checkmark
$ \begin{array}{c} \{d_k:d_j\},\\ \{d_k:\overline{d_j}\} \end{array} $			\checkmark	\checkmark
$\{ \overline{d_k} : d_j \}, \\ \{ \overline{d_k} : d_j \}, \{ d_k : \overline{d_j} \}$				\checkmark

Table 2: For i = 1, 2, 3, 4 the formula C_{ij} is the disjunction of the ticked expressions in the *i*-th row.

Lemma 5. The formula C_{ij} for each voter $i \in V$, and each issue $I_j \in I$, can be written in CNF with at most 2 clauses, and where each clause contains at most 2 literals.

Proof of Lemma 5. Fix a voter i and an issue I_j . For the cases where issue I_j has no inneighbor in G_i , the lemma obviously holds, as can be verified in Table 1. For all other cases, I_j has a unique in-neighbor, say issue I_k , since we are dealing with instances where the in-degree is at most one. We now need to examine the form of C_{ij} for the cases that arise.

Case A. If voter *i* is satisfied only with 1 out of the 4 possible outcomes regarding I_j and I_k , then C_{ij} is a disjunction of 3 conjunctions. Let us assume that C_{ij} is in the form: $(x_j \wedge x_k) \lor (\overline{x}_j \wedge x_k) \lor (x_j \wedge \overline{x}_k)$. All other cases are handled in exactly the same way. The following equivalences can bring C_{ij} to the desirable form.

$$(x_j \wedge x_k) \lor (\overline{x}_j \wedge x_k) \lor (x_j \wedge \overline{x}_k) \equiv x_k \lor (x_j \wedge \overline{x}_k) \equiv (x_k \lor x_j) \land (x_k \lor \overline{x}_k) \equiv x_k \lor x_j$$

Case B. If voter *i* is satisfied with 2 out of the 4 possible outcomes, then C_{ij} is a disjunction of 2 conjunctions. Without loss of generality, we can assume we have one of the following cases (all remaining cases can also be brought to one of these formats). As verified below, by the right hand side of each term, all cases can be brought into the desirable form.

(1.)
$$(x_j \wedge \overline{x}_k) \lor (\overline{x}_j \wedge x_k) \equiv (x_j \lor x_k) \land (\overline{x}_j \lor \overline{x}_k)$$

(2.)
$$(x_j \wedge x_k) \lor (\overline{x}_j \wedge \overline{x}_k) \equiv (x_j \lor \overline{x}_k) \land (\overline{x}_j \lor x_k)$$

Case C. If voter *i* is satisfied with 3 out of the 4 possible outcomes regarding I_j and I_k , then we simply take the conjunction expressing the outcome that causes dissatisfaction. E.g., $C_{ij} = x_k \wedge \overline{x}_j$, when *i* is satisfied with everything apart from $\{d_k : \overline{d_j}\}$. Thus, C_{ij} has 2 clauses with 1 literal each.

Note that, typically, according to the definition of the conditional approval framework, a voter could also be satisfied with all 4 possible outcomes of I_j and I_k or disagree with all possible outcomes. However, one can consider such ballots as simple unconditional ballots, as no real dependence between the issues exists.

Using Lemma 5 to convert each C_{ij} to CNF, we can now create a MIN 2-SAT instance P' by the multiset⁵ of all clauses appearing in the C_{ij} 's, i.e., appearing in the formula

$$C = \bigwedge_{i \in V, I_j \in I} C_{ij}.$$
 (1)

We now try to exploit how the analysis so far can help us for 1-approval instances. We will first need to compare the optimal solution of the CMS instance with the optimal solution of the corresponding MIN 2-SAT instance.

Lemma 6. Let P be a CMS instance and P' be its corresponding MIN 2-SAT instance produced as discussed in the proof of Theorem 4. Let also OPT(P) and OPT(P') be the values of the optimal solutions of the instances. If P is 1-approval, then it holds that OPT(P') = OPT(P).

Proof of Lemma 6. Consider an optimal solution of the CMS instance P. Every voter contributes to the cost of this solution precisely the number of issues with which she is dissatisfied. Consider now the corresponding MIN 2-SAT instance P', formed by the clauses of the constructed formula C from Equation (1). Let us look at the truth assignment to the variables of C, as dictated by the values of the issues in the optimal solution of P. We will provide an upper bound on the number of satisfied clauses of C. Under this truth assignment, it holds that for every voter i and for every issue I_i for which i is dissatisfied with respect to I_j , the formula C_{ij} is true. By Lemma 5, any C_{ij} has at most two clauses which could be satisfied when C_{ij} is true. But in the case when P is 1-approval, then any fixed voter i either votes unconditionally on I_j or her ballot belongs to Case A from the proof of Lemma 5. In both cases, C_{ij} is formed by a single clause. Hence, by looking at all the clauses of C that come from combinations (i, j), for which voter i is dissatisfied with respect to issue I_i , we get a number of satisfied clauses equal to OPT(P). Let us focus now on pairs (i, j), for which voter i is satisfied with respect to I_i . Then, the corresponding formula C_{ij} is false. If the ballot of voter i with respect to I_j is unconditional or if her ballot corresponds to Case A of Lemma 5, then C_{ij} does not have any true clauses. Therefore, for 1-approval instances, OPT(P'), which is the total number of satisfied clauses of C under the selected truth assignment, equals OPT(P). \square

Our construction gives rise to the following algorithm for CMS, under the discussed assumptions:

Algorithm 1	⊳Input: 1-approval profile P
-------------	------------------------------

1: Create P' from P using Lemma 5 and Equation (1).

- 2: Run an α -approximation of MIN 2-SAT on P'.
- 3: Set the value of I_j in P to the value of x_j in P'.

To conclude the proof of Theorem 4, let SOL(P') be the cost of the solution to P' produced in step 2 of Algorithm 1, which equals the number of satisfied clauses in C by

^{5.} Some clauses may happen to appear more than once in the final formula but there is no harm in keeping such duplicates.

the truth assignment of the α -approximation algorithm. This corresponds to a solution for CMS and let SOL(P) be its total cost. We note that the total number of distinct pairs (i, j) for which voter i is dissatisfied by issue I_j can be no more than the number of the satisfied clauses of C, since each C_{ij} corresponds to a pair of a voter and an issue. Hence, together with Lemma 6, we have the following implications:

$$SOL(P) \le SOL(P') \le \alpha \cdot OPT(P') = \alpha \cdot OPT(P)$$

Thus, every α -approximation algorithm for MIN 2-SAT yields an α -approximation for CMS, as long as P is 1-approval. To obtain the claimed approximation ratio, we use the algorithm for MIN 2-SAT from (Avidor & Zwick, 2005), which achieves a factor of 1.1037. \Box

Remark 2. The proof of Theorem 4 also reveals why we cannot extend it to have a constant approximation for other than 1-approval instances. In particular, for instances that involve the Cases B and C, described in the proof of Lemma 5, we cannot guarantee that Lemma 6 will hold (all we need is that $OPT(P') \leq OPT(P)$, but this could be far from true).

Although we have not been able to obtain a constant factor approximation for any other instance of CMS, the proof of Theorem 4 motivates the study of two more special cases of interest, for which we can obtain a positive result via different procedures. The central idea is that the general construction presented in Lemma 5 identifies 2 more cases, other than 1-approval, that may occur regarding the satisfaction of a voter w.r.t. an issue. These are precisely the Cases B and C in the proof of Lemma 5. To define these two cases more formally, once again, we consider instances with binary domains, and with $\Delta_i \leq 1$ for every voter i. We will provide positive results for the families of instances that will be called ORinstances and XOR-instances. The former family could be seen as a generalized variant of antiplurality instances (a.k.a. veto, see e.g. (Brandt, Conitzer, Endriss, Lang, & Procaccia, 2016) for more details); it contains instances in which every voter who casts a ballot for an issue I_i that is conditioned on the outcome of an issue I_k , approves exactly three out of the four possible combinations for these issues, or equivalently, is dissatisfied only with a single combination (the reason behind the name of this family will become clear when inspecting the proof of the approximate result that follows). The latter, includes instances in which for every issue I_j that is dependent on an issue I_k according to the preferences of a voter i, we assume that these issues are of a complementary nature, i.e., that voter i either wants I_i to be set to the same value as I_k or to the opposite (but not both). In other words, the satisfaction of the voter depends on the XOR value between I_i and I_k . In both families, we impose no restrictions for the issues that have no dependence on other issues. The formal definitions of the instances that we are going to examine follow.

Definition 7. We say that a CMS instance where the issues are binary and the dependency graph of every voter *i* satisfies $\Delta_i \leq 1$ is an OR-instance if every voter who casts a ballot on an issue I_j that is conditioned on the outcome of an issue I_k is approving all but one combinations $\{x_k : x_j\}$, where $x_\ell \in \{d_\ell, \overline{d_\ell}\}$, for $\ell \in \{k, j\}$.

Definition 8. We say that a CMS instance where the issues are binary and the dependency graph of every voter *i* satisfies $\Delta_i \leq 1$ is a XOR-instance if every voter who casts a ballot on an issue I_j that is conditioned on the outcome of an issue I_k , is voting either for $\{d_k : d_j, \overline{d_k} : \overline{d_j}\}$ or for $\{d_k : \overline{d_j}, \overline{d_k} : d_j\}$.

In contrast to the proof of Theorem 4, we are now going to reduce to and use algorithms for the MIN-2-CNF-DELETION problem for OR-instances and the MIN-UNCUT problem for XOR-instances. Likewise in (Agarwal, Charikar, Makarychev, & Makarychev, 2005), we are going to define these problems in a unified and convenient to us formulation, and we will consider them as special cases of the CONSTRAINT SATISFACTION PROBLEM (CSP) which also appeared in Section 3.1. Say that we are given a set of boolean variables b_1, \ldots, b_n and a set of constraints C and the goal is to find an assignment that minimizes the number of unsatisfied constraints. The MIN-2-CNF-DELETION problem is the special case of CON-STRAINT SATISFACTION PROBLEM in which each constraint can be written in a 2-CNF form. More precisely we will focus on instances in which each constraint corresponds to a single clause in 2-CNF form, which has been called 2-CNF CLAUSE-DELETION problem in the literature (Klein, Plotkin, Rao, & Tardos, 1997). The MIN-UNCUT problem is the special case of CONSTRAINT SATISFACTION PROBLEM in which each constraint is of the form $b_i \oplus b_j = 0$ or $b_i \oplus b_j = 1$ and has also been called 2-CNF \equiv DELETION in the literature (Garg, Vazirani, & Yannakakis, 1996).

Theorem 7. Let \mathcal{F} be the family of CMS instances where the issues are binary and the dependency graph of every voter *i* satisfies $\Delta_i \leq 1$. If \mathcal{F} only contains XOR-instances (resp. OR-instances), then any α -approximation algorithm for MIN-UNCUT (resp. 2-CNF CLAUSE-DELETION) yields an α -approximation algorithm for the family \mathcal{F} . In particular, we can have a polynomial time log m-approximation for the class of XOR-instances and a polynomial time log m-approximation for the class of OR-instances.

Proof. We start by proving the statement for XOR-instances. The result follows from an approximation preserving reduction of an instance $P \in \mathcal{F}$ to an instance P' of MIN-UNCUT. This reduction is similar to the one presented in the proof of Theorem 4 but, in this case, the satisfaction of a voter with respect to an issue should correspond to a satisfied constraint. Consider a voter i of P that has an unconditional ballot with respect to an issue I_i , say in favor of the alternative $x_j \in D_j$ (resp. $\overline{x_j} \in D_j$), then, her preference can be simply expressed as $x_j \oplus 0 = 1$ (resp. $x_j \oplus 1 = 1$). On the other hand, if the voter's ballot on I_j is conditioned on the outcome of I_k , due to the fact that P is a XOR-instance, her preferences can be expressed as $x_j \oplus x_k = 0$ or $x_j \oplus x_k = 1$. We have now created an instance P' of MIN-UNCUT, and, in analogy to the proof of Theorem 4, one can show that the costs of the optimal solutions of the two instances coincide. Similarly, it also holds that SOL(P) = SOL(P'), where SOL(P') corresponds to the cost of the solution of an α -approximation algorithm for MIN-UNCUT, whereas SOL(P) corresponds to the cost of the solution of the algorithm that transforms any XOR-instance P of CMS to an instance P'of MIN-UNCUT, as previously described, and then uses an α -approximation algorithm for MIN-UNCUT on P'. Utilizing the log n approximation from (Garg et al., 1996) (Section 8) therein) for MIN-UNCUT, where n is the number of variables in the instance, we obtain a $\log m$ approximation for CMS, under the discussed assumptions.

When it comes to OR-instances, it suffices to observe that a voter that casts a conditional ballot in such an instance could express his preference with a logical formula that is of the following form: $(x_j \wedge x_k) \lor (\overline{x}_j \wedge x_k) \lor (x_j \wedge \overline{x}_k)$, for some $x_j \in \{d_j, \overline{d_j}\}$ and $x_k \in \{d_k, \overline{d_k}\}$. But such an expression can be equivalently written as $x_k \lor (x_j \wedge \overline{x}_k)$ which, in turn, is equivalent to $x_k \lor x_j$, which can be seen as a constraint that is formed by a single clause in 2-CNF form. The rest of the arguments are identical to the case of XOR-instances and the approximation factor follows from (Klein et al., 1997) (Section 3.3 therein), where a $\log k \log \log k$ approximation algorithm is presented for the 2-CNF CLAUSE-DELETION problem, where k is the number of variables in the given formula.

We note that slightly better approximation factors are possible for CMS, under both assumptions, leveraging results from (Agarwal et al., 2005). However, this comes with the caveat of introducing randomization techniques, a debatable aspect in the context of social choice settings. Concluding this section, we highlight the attainment of a bounded approximation guarantee for every conceivable scenario, for the cases where all voters that are casting conditional ballots approve either one, two, or three combinations of values. Therefore, we have achieved positive results across the spectrum, albeit exclusively under the assumption that voters are required to approve the same number of combinations per conditional issue.

4. Strategic Control of CMS Elections

In this section, we consider strategic aspects of CMS and study questions related to controlling an election of interdependent issues, which falls under the broad and well studied umbrella of influencing election outcomes. Suppose that there is an external agent (called *controller*) who has a strong preference for a specific value of some (or every) issue in a CMS election. One of the instruments for enforcing a desirable value for the issue(s) the controller cares about, is by enabling new voters to participate or by disabling some existing voters, which can be done for example by changing the criteria for eligibility of voters. Furthermore, a controller could add more choices for the issues under consideration or delete existing ones, towards enforcing her will. We refer to (Chen, Faliszewski, Niedermeier, & Talmon, 2017) for related examples and further motivation. Finally, it is reasonable to assume that the controller does not have unlimited power, and therefore, she is capable of adding or deleting only a certain number of voters or alternatives.

Each combination of control features (i.e., addition vs deletion, voters vs alternatives, single issue vs multiple issues) gives rise to a different control type, namely control either all or a single issue by deleting voters (CDV), by adding voters (CAV), by deleting alternatives (CDA), or by adding alternatives (CAA). In this manner, we obtain 8 distinct algorithmic problems. Following the terminology of (Hemaspaandra et al., 2007), we say that a voting rule is *vulnerable* to a certain control type, if the corresponding problem is always solvable in polynomial time. If the problem is C-hard for a complexity class C, we consider the rule to be *resistant*⁶ to the specific control type (typically C is the class NP). In the cases where it is not possible for a controller to affect the election towards fulfilling her will, independent of complexity theory assumptions, we say that the rule is *immune* to the corresponding control type. The formal definitions of the control problems appear in the following subsections and are adaptations to CMS elections, of the original definitions of control problems provided in (Bartholdi III et al., 1992). An overview of the results we obtained follows.

^{6.} It's crucial to bear in mind that the notion of resistance is rooted in the realm of worst-case instances, supported by our NP-hardness results. These results indeed pose a barrier for controllers seeking to manipulate the outcomes, but this may not be true for every instance of the problem. In real-world scenarios, the susceptibility of CMS to such attempts may exhibit different behaviour.

	CDV & CAV		CDA			CAA			
					$\Delta = 1$				
	$d = \mathcal{O}(1)$	$d = \omega(1)$	$d = \mathcal{O}(1)$	$d = \Omega(1)$	$d = \mathcal{O}(1)$	$d = \Omega(n)$	$d = \Omega(1)$	$d = \Omega(n)$	$d = \mathcal{O}(1)$
ALL	R	R	R	V	R	R	Ι	Ι	Ι
1	V	R	R	V	V	R	Ι	R	R

Table 3: Results on Controlling CMS elections. R stands for (worst-case) Resistant (i.e. NP-hard), V for Vulnerable (i.e. polynomially solvable) and I for Immune (i.e. impossible, independent of complexity theory assumptions). For a CMS instance on n voters, we denote as Δ the maximum in-degree of every voter's dependency graph ($\Delta = \max_{i \in [n]} \Delta_i$) and d the maximum domain size.

4.1 Controlling Voters

We start with the problems of adding or deleting voters for enforcing a specific outcome either for a single issue or for every issue of the election.

Instance: A CMS election (I, D, V, B), where V is the set of registered voters, a set V' of yet unregistered voters with $V \cap V' = \emptyset$ (for use only by CAV), an integer quota q, a distinguished alternative $p_j \in D_j$ for a specific issue I_j or an outcome $p \in D$ (for the "ALL" versions) specifying an alternative for every issue.

Problem CAV-1 (resp. CDV-1): Does there exist a set $V'' \subseteq V'$ (resp. $V'' \subseteq V$), with $|V''| \leq q$, such that p_j is the value of issue I_j in every optimal CMS solution of the profile $(I, D, V \cup V'', B)$ (resp. of the profile $(I, D, V \setminus V'', B)$)?

Problem CAV-ALL (resp. CDV-ALL): Does there exist a set $V'' \subseteq V'$, (resp. $V'' \subseteq V$) with $|V''| \leq q$, such that p is the unique optimal CMS solution of the profile $(I, D, V \setminus V'', B)$ (resp. of the profile $(I, D, V \setminus V'', B)$)?

Remark 3. One has the option of either breaking ties in favor of the controller, if there are multiple optimal solutions in CMS (as in (Davies, Katsirelos, Narodytska, & Walsh, 2011)), or demand that the controller's will is fulfilled in every optimal outcome. We focus on the second case, as is also done in the seminal paper of Bartholdi et al. (Bartholdi III et al., 1992). Additionally, it is possible that the controller has a strong opinion not just for a single or all issues, but for a subset of issues. As a starting point, we have chosen to consider the two extremes (and intuitively simpler versions).

We now present our results for these 4 problems, exhibiting that it is not generally easy for a controller to enforce her will in such elections. In fact, computational hardness of controlling by adding or deleting voters can be established even for very simple forms of elections, without even the presence of conditional ballots, as shown in the two theorems that follow.

Theorem 8. CDV-ALL is NP-hard even for Unconditional Minisum and for a binary domain in each issue.

Proof. To prove the NP-hardness, we will have a reduction from the VERTEX COVER problem. Thus we start with an instance (G = (V, E), k), which asks if there is a vertex cover of size at most k, and create an instance P of CDV-ALL.

For every edge $e \in E$, we add an issue I_e having two possible alternatives, and denote its domain by $D_e = \{d_e, \overline{d_e}\}$. For every vertex $v \in V$, we add a voter voting unconditionally for d_e , if e is incident to v and being satisfied with both $\{d_e, \overline{d_e}\}$ otherwise. Let there also be 2 dummy voters who are satisfied only with $\overline{d_e}$ for every issue I_e . Hence, all the ballots are unconditional, and we have an empty global dependency graph. For the quota parameter, we use q = k, and suppose that the controller wants to enforce the alternative $\overline{d_e}$ for every issue I_e . This completes the description of the CDV-ALL instance, where the goal is to decide if there exists a set V'' of size at most q, such that deleting those voters enforces the controller's desirable outcome.

Suppose that there exists a vertex cover $S \subseteq V$ of G, of size at most k. Since each edge of G has at least one endpoint in S, by removing all voters that correspond to S, each alternative d_e loses at least one approval vote. Hence, d_e would cause two dissatisfactions to the dummy voters (the others are indifferent), whereas $\overline{d_e}$ causes at most one dissatisfaction. Therefore, selecting the alternative $\overline{d_e}$ for every issue I_e is the unique optimal solution,

For the reverse direction, suppose there exists a set of voters S, whose removal causes the outcome $(\overline{d_e})_{e \in E}$ to become the unique optimal solution. First, we may assume that Sdoes not contain any of the dummy voters (otherwise, add them back to the instance, and the total dissatisfaction score will not be affected). Suppose that S is not a vertex cover in G, and that at least one edge e is not covered by S. But this means that the removal of S from the CDV-ALL instance will leave intact the two voters that are satisfied only with d_e , and therefore d_e can also be selected in an optimal solution (it causes the same number of dissatisfactions as $\overline{d_e}$). This contradicts the fact that the removal of S resulted in the unique optimal solution with $\overline{d_e}$ selected for every issue I_e .

Theorem 9. CAV-ALL is NP-hard even for Unconditional Minisum and for a binary domain in each issue.

Proof. The proof is a simple adaptation of a reduction given for almost the same problem but in the context of the classic (unconditional) approval voting rule in (Hemaspaandra et al., 2007). For the sake of completeness, we provide the full construction here. We stress that we cannot directly establish NP-hardness by applying the result of that work because when there are no conditional ballots, the version of approval voting as defined there selects as winner(s) the candidates who have the highest number of approvals, whereas Unconditional Minisum selects only candidates who are approved by at least 50% of the voters. In the instances used in the reductions of (Hemaspaandra et al., 2007) (see Theorem 4.43 therein), there are losing candidates who are approved by more than 50% of the voters, hence their proofs do not apply directly.

We start with an instance P of EXACT-3-COVER (X3C) where $B = \{b_1, \ldots, b_m\}$ with m = 3k is the universe, and $\mathcal{F} = \{S_1, \ldots, S_n\}$ is a collection of sets with $|S_i| = 3$, for every set S_i . The goal is to decide if there is an exact cover, i.e. a subcollection of sets from \mathcal{F} such that each element of the universe belongs to exactly one of these sets.

We now define a CMS election where the set of issues is $I = B \cup \{I_{m+1}\}$ and each issue has a binary domain, with $D_j = \{b_j, \overline{b_j}\}$ for $j \in [m]$, and $D_{m+1} = \{w, \overline{w}\}$. The set of voters is as follows:

- There are k-2 registered voters who are satisfied with b_j for $j \in [m]$, and with \overline{w} . They are dissatisfied with the complements of these alternatives.
- There is one registered voter who is satisfied only with $\overline{b_j}$ for $j \in [m]$ and with \overline{w} .

• There are *n* unregistered voters corresponding to the sets of x3c instance. The voter corresponding to S_i is satisfied only with the 3 alternatives of S_i , and with *w*.

To finish the description, we set the quota parameter q equal to k and the desirable outcome of the controller to be $(\overline{b_1}, \ldots, \overline{b_m}, w)$. Hence, the goal in the CAV-ALL instance is to decide if there exists a set of unregistered voters V'' with $|V''| \leq k$ such that adding V'' to the registered voters makes the desirable outcome the unique optimal solution.

Suppose now that there exists an exact cover in P. Since m = 3k, the cover consists of exactly k sets. Select as V'' the k unregistered voters corresponding to the cover. We now have a total of 2k-1 voters in the election. For the first m issues, the alternative b_j satisfies exactly k-1 voters and dissatisfies k voters, hence the optimal solution selects $\overline{b_j}$ for $j \in [m]$. For the last issue, the value w satisfies k voters and dissatisfies the remaining k-1 voters. Hence, the unique optimal solution when adding the set V'' is precisely $(\overline{b_1}, \ldots, \overline{b_m}, w)$.

For the opposite direction, suppose that there is a set V'' of unregistered voters, with $|V''| \leq k$, such that when adding them to the registered voters, the unique optimal solution is the controller's desirable outcome. First notice that this implies that |V''| = k, otherwise there is not enough support for w to be selected. The only other possibility would be to have |V''| = k - 1, but then we have a tie, and there would be more optimal solutions with \overline{w} instead of w. Since for the other issues, each b_j already has a support by k - 2 registered voters, then none of them received a support by two or more of the added voters. But these voters express a support for a total of 3k = m such alternatives, therefore, each b_j for $j \in [m]$, receives support by exactly one of the added voters.

The next step is to see whether the hardness results of Theorems 8 and 9 go through when the controller wishes to control just a single issue. For Unconditional Minisum this is not the case if we insist on a constant domain size for the designated issue. The reason is that this can be reduced to an FPT version of the well known Set MultiCover problem.

Proposition 1 (implied by (Bredereck, Faliszewski, Niedermeier, Skowron, & Talmon, 2020)). CAV-1 and CDV-1 can be solved in polynomial time for Unconditional Minisum if the domain size of each issue is constant.

As a consequence, any potential hardness result for CAV-1 and CDV-1 would have to consider either non-constant domain sizes or conditional ballots. Indeed, we establish that either of these settings suffices to establish NP-hardness. We start with elections where at least one issue has a non-constant domain size.

Theorem 10. CAV-1 and CDV-1 are NP-hard, even for Unconditional Minisum, but with non-constant domain size in at least one issue.

Proof. We will only describe the proof of NP-hardness for CAV-1 and the same can be established for CDV-1 in a very similar fashion, using almost the same reduction.

We will have a reduction from the problem of controlling a classic approval voting election by adding voters, proved NP-hard in (Hemaspaandra et al., 2007). We recall that in an approval voting election, voters express their approved set of candidates, and the winner (or winners in case of ties) is the candidate with the highest number of approvals. The control problem there is to ensure that a designated candidate is the unique winner of the election. Our reduction starts with an instance P of the control problem in approval voting, where V and V' are the registered and unregistered sets of voters respectively, p is a designated candidate, and q is a quota. The goal is to select a set $V'' \subseteq V'$ with $|V''| \leq q$, so that the approval voting rule, when run on the voters in $V \cup V''$ will select p as the unique winner.

We create an instance P' of CAV-1 where the sets of voters, registered and unregistered, are the same as in P. If the number of candidates in P is m, we create a single issue in P' whose domain has exactly m possible alternatives, and p is the designated alternative that the controller wants to promote in P'. For every voter in P (whether coming from Vor V'), the corresponding voter in P' specifies an unconditional ballot on the single issue, containing only her approved options in P. We also use the same quota parameter q as in P. This completes the description of P', which can be clearly constructed in polynomial time.

It is now easy to see that there exists a set $V'' \subseteq V'$ of at most q voters so as to ensure that p will be the outcome on the single issue of P', using the CMS rule for the voters of $V \cup V''$, if and only if the same set of voters can ensure that p will be the unique winner in the approval voting election of P. Indeed, if the CMS rule, run on the voters of $V \cup V''$, selects the outcome p in the instance P', this means by the definition of the CMS rule that pcauses the minimum number of dissatisfactions among all possible alternatives, i.e., it has the highest number of approvals. This directly yields that p will be the unique winner in the instance P. The reverse direction is easy to see as well, with the same reasoning. \Box

We now study the hardness of these problems when we have conditional ballots. As the next theorem shows, it suffices to consider only profiles where each issue may depend on at most one other issue.

Theorem 11. CDV-1 and CAV-1 are NP-hard, when $\Delta \leq 1$, even for a binary domain in every issue.

Proof. We will focus on proving the statement for CDV-1. The proof for CAV-1 is very similar, requiring minimal changes, that we discuss at the end. We will prove that CDV-1 with a binary domain for every issue, and with $\Delta = 1$ is NP-hard, using a reduction from the NP-hard version of CDV-1 with $\Delta = 0$ and non-constant domains (Theorem 10). We firstly remind the reader that the proof of Theorem 10 indicates that the problem CDV-1 with $\Delta = 0$ and non-constant domain sizes is NP-hard, even for the family of instances with just a single issue (that has a non-constant number m of different alternatives), where every voter casts only approval ballots for a subset of alternatives.

Our reduction starts from an instance P of CDV-1 on a single issue with non-constant domain size (obviously $\Delta = 0$, since we have only one issue), and creates an instance P'of CDV-1 with binary domain for every issue and with $\Delta = 1$. Say that the issue of P has the following m different alternatives: $\{d_1, d_2, \ldots, d_m\}$. Then the instance P' consists of mdifferent binary issues I_1, I_2, \ldots, I_m such that the alternatives of issue I_j are $\{d_j, \overline{d_j}\}$, for $j \in [m]$. Hence, the idea is that each alternative of the single issue of P now corresponds to a different issue in P' with positive and negative alternatives. Furthermore, if q was the quota in P, we will use the same quota in P'. Finally, if d_ℓ was the designated alternative in P, for some specific $\ell \leq m$, then we will have that d_ℓ is the designated alternative for issue I_ℓ in P'. We describe now the set of voters in P' as well as their preferences. For every voter v of P, we add a voter v' in P', such that for any $j \in [m]$, if v was approving the alternative d_j for the single issue of P, then v' approves d_j concerning the issue I_j in P', otherwise, if v was not approving d_j in P, then v' is indifferent in P' and votes for $\{d_j, \overline{d_j}\}$. We also add a set of m(m-1)L dummy voters, where L = nm + q + 1 and n is the number of voters in P. In particular, for every ordered pair of distinct alternatives of P, i.e., for every (k, j), with $k, j \in [m]$, and $k \neq j$, we include L voters voting $\overline{d_k}$ for issue I_k , $\{d_k : \overline{d_j}\}$ for issue I_j , and $\{d_t, \overline{d_t}\}$, for every other issue I_t , with $t \in [m] \setminus \{k, j\}$.

We will first prove the following claim, which is enforced by the construction of P'.

Claim 4. Let P'' be the conditional approval voting profile that is derived by the deletion of a set of at most q non-dummy voters from the instance P'. Then, in any optimal CMS solution of P'', there is exactly one issue I_j for which the selected alternative is d_j , and for every $k \neq j$, the selected alternative will be $\overline{d_k}$.

Proof of Claim 4. Let us define first the set of outcomes where exactly one issue takes a positive value, i.e., let $POS_1 = \{\{y_1, y_2, \ldots, y_m\} : \exists i \in [m] : y_i = d_i, \forall j \in [m] \setminus \{i\} : y_j = \overline{d_j}\}$. We will firstly prove that for any solution that belongs to POS_1 , the total dissatisfaction incurred by the set of dummy voters equals m(m-1)L. To prove that, we inspect an arbitrary outcome of POS_1 , say $\{\overline{d_1}, \overline{d_2}, \ldots, \overline{d_{p-1}}, d_p, \overline{d_{p+1}}, \ldots, \overline{d_m}\}$, for some $p \in [m]$. It is convenient to view the set of m(m-1)L dummy voters of P'', as being partitioned in the following 3 sets:

- The (m-1)(m-2)L dummy voters whose dependency graph consists of an edge (I_k, I_j) , such that $k, j \neq p$. These voters are dissatisfied only with respect to issue I_j since they are voting for $\{d_k : \overline{d_j}\}$ and d_k is not selected. They are satisfied with respect to I_k since they are voting in favor of $\overline{d_k}$, which is elected, and they are indifferent (hence satisfied) with respect to all other issues.
- The (m-1)L dummy voters whose dependency graph consists of an edge (I_p, I_j) , such that $j \neq p$. These voters are dissatisfied only with respect to issue I_p since they are voting for $\overline{d_p}$ but d_p is elected. They are satisfied with respect to I_j since they are voting in favor of $\{d_p : \overline{d_j}\}$ and both d_p and $\overline{d_j}$ are elected. Finally they are satisfied with respect to any I_t , for $t \neq p, j$, since they are indifferent for these issues.
- The (m-1)L dummy voters whose dependency graph consists of an edge (I_j, I_p) such that $j \neq p$. These voters are dissatisfied only with respect to issue I_p since they are voting for $\{d_j : \overline{d_p}\}$. Furthermore they are satisfied with respect to I_j since they are voting in favor of $\overline{d_j}$, and they are satisfied with respect to any other issue.

Hence, the total dissatisfaction score of any outcome in POS_1 , due to the dummy voters is m(m-1)L. To count the dissatisfaction from the remaining non-dummy voters, we define the following quantity: let x_i be the number of voters in the original instance P (after deleting the voters that correspond to the ones deleted in P') who had d_i in their approval list in P. Then, it can be verified that the total dissatisfaction due to the non-dummy voters for the profile P'', is $\sum_{i \in [m] \setminus p} x_i$. Hence, to concude, the total dissatisfaction score for P'', under any outcome that belongs to POS_1 is equal to $m(m-1)L + \sum_{i \in [m] \setminus p} x_i$.

We will now compare the dissatisfaction score of the outcomes in POS_1 with all other possible outcomes. We can define analogously the set of outcomes $POS_2 = \{(y_1, y_2, \ldots, y_m) : \exists i, j \in [m] : y_i = d_i, y_j = d_j, \forall k \in [m] \setminus \{i, j\} : y_k = \overline{d_k}\}$. We will firstly prove that the total dissatisfaction incurred by dummy voters in outcomes from POS_2 is at least m(m-1)L+2L. To prove it, we inspect an arbitrary outcome of POS_2 , say (y_1, \ldots, y_m) with $y_p = d_p$, $y_r = d_r$, for some specific p, r, and also $y_j = \overline{d_j}$, for any $j \neq p, r$. We analyze the set of dummy voters, by partitioning them in 4 sets as follows:

- The (m-2)(m-3)L dummy voters whose dependency graph has the edge (I_k, I_j) , for some k, j where both $k, j \neq p, r$. These voters are dissatisfied only with respect to issue I_j since they are voting for $\{d_k : \overline{d_j}\}$ but $\overline{d_k}$ is elected. They are satisfied with respect to I_k since they are voting in favor of $\overline{d_k}$. Finally they are indifferent and hence satisfied with respect to all other issues.
- The 2(m-2)L dummy voters whose dependency graph has either the edge (I_p, I_j) or (I_r, I_j)), with $j \neq p, r$. We analyze the ones with the edge (I_p, I_j) , and the same conclusion holds for the other case as well. These voters are dissatisfied only with respect to issue I_p since they are voting for $\overline{d_p}$, but d_p is elected. They are satisfied with respect to I_j since they are voting in favor of $\{d_p : \overline{d_j}\}$ and both d_p and $\overline{d_j}$ are elected. Finally they are satisfied with respect to any I_t , for $t \neq p, j$, since they are indifferent.
- The 2(m-2)L dummy voters whose dependency graph consists of the edge (I_j, I_p) or (I_i, I_r) for some $j \neq p, r$. We argue about the voters with the edge (I_j, I_p) as the other case is also identical. These voters are dissatisfied only with respect to issue I_p , since they are voting for $\{d_j : \overline{d_p}\}$ but d_p is elected. They are satisfied with respect to I_j since they are voting in favor of $\overline{d_j}$ which is elected. Finally they are indifferent with respect to other issues.
- The 2L dummy voters whose dependency graph consists of the edge (I_p, I_r) or (I_r, I_p) . These voters are dissatisfied with respect to issues I_r and I_p . Consider the ones with the edge (I_p, I_r) . They are voting for $\{d_p : \overline{d_r}\}$ but d_r is elected. Additionally, they are voting for $\overline{d_p}$ but d_p is elected. They are satisfied with respect to any other issue since they are indifferent.

The above analysis shows that the dissatisfaction score due to the dummy voters is (m-2)(m-3)L + 2(m-2)L + 2(m-2)L + 4L = m(m-1)L + 2L. This is larger by the term 2L, compared to the dissatisfaction of the dummy voters in outcomes of POS_1 . By the choice of L, it is impossible that the dissatisfaction of the non-dummy voters causes an outcome of POS_2 to have a better or equal score than those of POS_1 (note that L > mn and the total dissatisfaction of the non-dummy voters is bounded by mn). Hence, the optimal solution cannot be attained by POS_2 . In a similar manner, we can define POS_k , for any k > 2, and prove that the total dissatisfaction score of any outcome in POS_1 is less than the dissatisfaction score of any outcome in POS_k . Equivalently, there can be no more than a single positive alternative in the optimal outcome of P''.

Finally, we also need to compare with the "all-negatives" outcome, where $\overline{d_j}$ is selected for every $j \in [m]$. It is a matter of a simple case analysis (like the ones before) to prove that the dissatisfaction incurred by the dummy voters equals m(m-1)L, which equals the dissatisfaction of the dummy voters for outcomes in POS_1 as well. Hence, now we only need to argue about the non-dummy voters; it is safe to say that these are more than q. To do this, note that in the original instance P, we can assume without loss of generality that at least one voter approves at least one alternative, otherwise we have a trivial election, where no one expresses any preferences. This means that $x_p > 0$ for at least one issue $p \in [m]$. By the construction of the preferences for the non-dummy voters in P', the dissatisfaction score for the "all-negatives" outcome equals $\sum_{i \in [m]} x_i$. Recall also that for the outcome in POS_1 , where only issue p has a positive value, the dissatisfaction of the non-dummy voters is $\sum_{i \in [m] \setminus \{p\}} x_i$. Therefore, we have that there exists an outcome in POS_1 which causes strictly less dissatisfaction to the electorate, and this concludes the proof.

Let P be a yes instance, i.e., say that there is a set S of at most q voters, the deletion of which causes the election of d_{ℓ} in P, as the unique winner. Note that for every voter in P there is a corresponding voter in P'. Consider the deletion of the set S' from P' that corresponds exactly to the voters of S from P. By exploiting Claim 4, it suffices to prove that by deleting S', the outcome $p_{\ell} = \{\overline{d_1}, \overline{d_2}, \ldots, \overline{d_{\ell-1}}, d_\ell, \overline{d_{\ell+1}}, \ldots, \overline{d_m}\}$ causes strictly less dissatisfactions in the electorate than $p_j = \{\overline{d_1}, \overline{d_2}, \ldots, \overline{d_{j-1}}, d_j, \overline{d_{j+1}}, \ldots, \overline{d_m}\}$, for any $j \neq \ell$. To do so, for any $i \in [m]$, let x_i be the number of voters (among the remaining ones, after deleting the set S) who had d_i in their approval list in P. By using the same argument as in the proof of Claim 4, we can see that for the non-dummy voters of P' who correspond to those of P, the number of dissatisfactions caused by the outcome p_{ℓ} in P' is $(\sum_{i \in [m] \setminus \ell, j} x_i) + x_j$. Also, by doing the same counting argument as in the first part of the proof of Claim 4, each dummy voter will be dissatisfied with exactly one issue, and hence they contribute a total of m(m-1)L in the cumulative number of dissatisfactions. In the same manner, we can also have that the total number of dissatisfactions caused by the outcome p_j is $(\sum_{i \in [m] \setminus \{l,j\}} x_i) + x_\ell + m(m-1)L$. Given that d_ℓ was the winning alternative in P after the deletion of voters, it is true that $x_{\ell} > x_j$ and hence the dissatisfaction caused by p_i is greater than the dissatisfaction caused by d_{ℓ} . By Claim 4, this concludes the first direction of the proof.

For the reverse direction, say that there is a set of voters S' in P', after the deletion of which, d_{ℓ} is the winning alternative for issue I_{ℓ} . We note that because of the large value of L, including any dummy voter in the set S' will not influence the final outcome (since L > q, even if all the deleted voters are dummy ones, we cannot enforce a different outcome than the outcome without deletions). Hence, we can assume that the set S' has no dummy voters. We can choose then to delete the corresponding set of voters S in Pand we will need to prove that, in that case, the elected alternative will be d_{ℓ} for the single issue of P. By Claim 4, and since we assumed that d_{ℓ} is selected for issue I_{ℓ} , we know that the optimal solution in the instance after the deletion of S' selected $\overline{d_i}$ for any other $j \neq \ell$. Thus, the winning outcome is $p_{\ell} = \{\overline{d_1}, \overline{d_2}, \ldots, \overline{d_{\ell-1}}, d_{\ell}, \overline{d_{\ell+1}}, \ldots, \overline{d_m}\}$. Therefore, the number of dissatisfactions caused by p_{ℓ} is lower than the number of dissatisfactions caused by $p_j = \{\overline{d_1}, \overline{d_2}, \ldots, \overline{d_{j-1}}, d_j, \overline{d_{j+1}}, \ldots, \overline{d_m}\}$, for any $j \neq \ell$. But this implies that $(\sum_{i \in [m] \setminus \{\ell, j\}} x_i) + x_j < (\sum_{i \in [m] \setminus \{\ell, j\}} x_i) + x_\ell$, or equivalently, $x_\ell > x_j$, which means that the number of voters that are not included in S, and have d_{ℓ} in their approval list, is greater than the number of voters not included in S, who have d_j in their list for any $j \neq \ell$. Thus, in P, d_{ℓ} will be selected, fulfilling the controller's will.

Adjustments for the CAV-1 reduction: So far we have established that CDV-1 is NP-hard. To prove that CAV-1 is also NP-hard for binary domains, we will perform a similar reduction, this time from CAV-1 with a single issue and with a non-constant domain size (which is again NP-hard by Theorem 10). The construction is almost the same, with the difference being that in CAV-1, there are both registered and unregistered voters. Our reduction will assign preferences in the same way as in the proof for CDV-1 and will simply maintain the separation into registered and unregistered voters in the created instance P'. Furthermore, we include the dummy voters in the set of registered voters and we set L = m(n+q) + 1. Everything else in the reduction is the same as before, and it is a matter of calculations analogous to the proof of CDV-1 to verify the correctness of the reduction.

To conclude this subsection, we have now a complete picture for the level of robustness against the malicious actions of adding or deleting voters. Our results act in favor of the CMS rule, showing that a potential controller cannot easily (in terms of computational complexity) enforce her own desirable outcomes, apart from a single case, as shown in Table 3. Interestingly, this, single, polynomially solvable case concerns the unconditional setting and when one moves to the conditional case, the considered problem becomes hard for the controller. Finally, it's worth noting that despite the need for distinct proof approaches in certain cases, the results for the two examined problems (deleting/adding voters) are identical; this observation does not extend to the problems related to the deletion and addition of alternatives, as elucidated in the forthcoming subsection.

4.2 Controlling Alternatives

We now consider the analogous control problems, regarding the addition or deletion of alternatives, instead of voters. It turns out that the picture, from the computational complexity viewpoint, differs sufficiently from the problems considered in the previous subsection.

Instance: A CMS election (I, D, V, B), where $D = D_1 \times \cdots \times D_m$, and D_k is the set of qualified alternatives of each issue I_k , a set D'_k of spoiler alternatives for each I_k (for use only by CAA), an integer quota q, a distinguished alternative $p_j \in D_j$ for a specific issue I_j or an outcome $p \in D$ specifying an alternative for every issue.

Problem CAA-1 (resp. CDA-1): Does there exist a set $D'' \subseteq \bigcup_{k \in [m]} D'_k$ (resp. $D'' \subset \bigcup_{k \in [m]} D_k$), with $|D''| \leq q$, such that p_j is the value of the issue I_j in every optimal CMS solution of the profile where the domain of each issue I_k is enlarged by the alternatives in $D'' \cap D'_k$ (resp. reduced by the alternatives in $D'' \cap D'_k$)?

Problem CDA-ALL: Does there exist a set $D'' \subset \bigcup_{k \in [m]} D_k$, with $|D''| \leq q$, such that p is the unique optimal CMS solution of the profile where the domain of each issue I_k is reduced by the alternatives in $D'' \cap D_k$?

Note: For CDA-1 and CDA-ALL, we also require that for every k, $|D_k \setminus D''| \ge 1$.

Remark 4. We firstly note that all the comments made in Remark 3 are applicable here as well. Also, we have not included CAA-ALL in our definitions as CMS is trivially immune to adding spoiler alternatives in order to enforce a qualified alternative in every issue. Concerning the problem CAA-1, we assume that the voters in B may express an opinion about any outcome of every issue, whether it is a qualified one or a spoiler. Additionally, another way to define such problems would be to allow the controller to completely delete or

add issues instead of just alternatives. However, given the existence of dependency graphs, erasing an issue can make the preferences of a voter ill-defined. Lastly, the constraint that $|D_k \setminus D''| \ge 1$, for CDA-1 and CDA-ALL, is to ensure that the controller cannot eliminate all the alternatives of an issue.

Proposition 2. Unconditional Minisum, with arbitrary domain size is immune to CAA-1. For the same setting, CDA-1 and CDA-ALL can be solved in polynomial time.

Proof. To solve CDA-1 and CDA-ALL we only have to observe that to control a single issue by deleting alternatives in the unconditional case, one can check if the quota is large enough to delete all alternatives that achieve higher approval score than the designated one(s). At what concerns CAA-1, the definition of immunity directly applies, since the controller cannot enforce a designated alternative in Unconditional Minisum by adding some other alternative (whether for the same or for a different issue).

When we move to instances with conditional ballots, the problems CDA-1 and CDA-ALL do become hard (with the exception of Proposition 3 in the sequel). We start with the hardness of CDA-ALL.

Theorem 12. CDA-ALL is NP-hard, when $\Delta_i \leq 1$ for every voter *i*, and even for a binary domain size in every issue.

Proof. Let P = (G, k) be an instance of VERTEX COVER, for an undirected graph G with n vertices and m edges, and an upper bound k on the desirable size of the cover. We denote by e_1, e_2, \ldots, e_m , the edges of the graph. We will present a reduction from P to an instance P' of CDA-ALL. Let there be the following 2m(k+1) + n issues:

- For every edge e_{ℓ} of G, $\ell \in [m]$, and for $j \in [k+1]$ let $I_{\ell,j}$ and $I'_{\ell,j}$ be a pair of binary issues of domain $\{e_{\ell,j}, \overline{e_{\ell,j}}\}$ and $\{e'_{\ell,j}, \overline{e'_{\ell,j}}\}$ respectively. We refer to these issues as edge issues.
- For every vertex v of G let I_v be a binary issue of domain $\{d_v, \overline{d_v}\}$. We refer to these issues as vertex issues.

Note that, we have added one vertex issue for every vertex of G and, essentially, we have introduced two edge issues for every edge of G, but in (k+1) 'copies'. This will play a significant role in the reverse direction of the reduction. The voters of the created instance P' will be 2m + 3 in total, and we partition them as follows:

- Group 1: There are 2 edge voters for every edge e_i of G. For an edge $e_i = (u, v)$ and for every $j \in [k+1]$, these voters submit the same ballot $\{d_u : e_{i,j}, d_v : e'_{i,j}\}$ (where we have arbitrarily chosen that the satisfaction of one edge issue depends on the vertex issue I_u and the other edge issue depends on the remaining vertex issue I_v). The voters are indifferent for the rest of the issues. Hence, every such voter is interested in exactly 2(k+1) edge issues, each conditioned on either I_u or I_v . At what follows, we will refer to these voters as "group 1" or edge voters.
- Group 2: There are 2 voters who are voting for $\{e_{i,j} : e'_{i,j}\}$, for every pair $(i, j) \in [m] \times [k+1]$, and who are indifferent for every other issue. At what follows, we will refer to these voters as "group 2".

• There is one voter, that we will refer to as the special voter, who is voting unconditionally $\overline{d_1}, \overline{d_2}, \ldots, \overline{d_n}$ for the vertex issues, and is indifferent for every other issue.

To complete the construction of the instance P', we use k as the quota parameter, and we suppose the controller wants to enforce the outcome $\{(\overline{e_{i,j}}, \overline{e'_{i,j}})_{i \in [m], j \in [k+1]}, (\overline{d_j})_{j \in [n]}\}$ by removing at most k alternatives. It is trivial to observe that all the issues in the instance P' are of binary domain and that for every voter i, $\Delta_i \leq 1$ in her dependency graph.

Before we proceed, we note that the designated outcome fully satisfies the voters from group 2 as well as the special voter, however, it dissatisfies every edge voter with respect to all 2(k + 1) issues they care about each, hence it produces a total dissatisfaction score of 4m(k + 1). Additionally, we highlight that there are also other outcomes with the same score, which prevent the designated one from being the unique winner without deleting any alternatives in P'. For instance, if e_t is the edge (u, v) of G, then consider the following outcome, for any $\ell \in [k + 1]$:

$$\{(e_{t,\ell}, e'_{t,\ell}), (\overline{e_{i,j}}, \overline{e'_{i,j}})_{(i,j)\in[m]\times[k+1]\setminus(t,\ell)}, du, dv, (\overline{d_j})_{j\in[n]\setminus\{u,v\}}\}$$

$$(2)$$

The outcome described in Equation 2 has a dissatisfaction score of 4(m-1)(k+1) + 4kfrom group 1 and a dissatisfaction score of 1 from each voter in group 2 and a dissatisfaction score of 2 for each remaining voter, leading to a total score of 4m(k+1).

To see now the forward direction of the reduction, suppose there exists a vertex cover in G of size at most k, which is formed say by a set of vertices $S \subset [n]$. Then we choose to delete in the created instance P' of CDA-ALL, the corresponding positive alternatives $\{d_j\}$ for $j \in S$, from the vertex issues $\{I_j\}_{j \in S}$, respectively. Hence, any solution to the resulting instance after these deletions, should now definitely contain the alternative $\overline{d_j}$ for every issue I_j , such that $j \in S$. Clearly, the designated outcome still remains a valid solution with a total score of 4m(k+1). We need to see what happens with the rest of the possible outcomes.

One can easily verify that the best solution among the feasible ones in which all edge voters are dissatisfied with respect to all the issues they care about, is precisely the designated outcome (because it satisfies in all issues all the other voters from group 2 as well as the special voter, and also no other solution can achieve the same). Hence, it remains to see if there exists any optimal solution with at least one pair of edge voters satisfied with respect to at least one issue.

For the sake of contradiction, assume that this is the case, and consider a pair of such edge voters, corresponding to edge e = (u, v). Without loss of generality, suppose that u does not belong to the vertex cover, and hence d_u is an available alternative, that has been selected in the optimal solution we are considering, and also that d_v has been deleted. This is due to the vertex cover property, implying that either d_u or d_v has been deleted and furthermore, if both had been deleted, the pair of edge voters that we focus on, would not be satisfied with respect to any issue, contrary to what we have assumed. The selection of d_u may also cause other edge voters (whose edge is incident with u) to be satisfied as well, with respect to some issues. To arrive at a contradiction, and for ease of notation, let e_1, \ldots, e_ℓ , be the edges that have u as an endpoint, and that correspond to edge voters who are satisfied with respect to at least one of the issues they care about, under the optimal solution we are considering. Additionally, for such an edge e_i , let m(i) be the number

of issues that correspond to e_i , and with respect to which, the two edge voters of e_i are satisfied (this occurs because the edge voters may have declared $d_u : e_{i,j}$ or $d_u : e'_{i,j}$). Clearly $m(i) \leq k+1$. Counting the total number of satisfactions of edge voters due to the selection of d_u , we get exactly $2\sum_{i=1}^{\ell} m(i)$. But if we now replace d_u by $\overline{d_u}$ and set all the edge issues of the edges e_1, \ldots, e_{ℓ} , to their negative value, then we will dissatisfy all the edge voters that we were satisfying before, but we get $2\sum_{i=1}^{\ell} m(i)$ new satisfactions from group 2 and one new satisfaction from the special voter. Therefore, we reach an outcome with a lower dissatisfaction score, contradicting the fact that we started with an optimal solution.

We conclude that there cannot exist an optimal solution, after the deletions we made, where some of the voters from group 1 enjoy any satisfaction. Therefore, we can only satisfy group 2 and the special voter, and we can conclude that after the deletion of at most k(positive) alternatives that correspond to a vertex cover, the designated outcome becomes the unique winner.

For the reverse direction, suppose that there is a set D of at most k alternatives, the deletion of which, forces the designated outcome to be the unique optimal solution. Trivially, D cannot contain negative values neither from edge nor from vertex issues. We denote by D_V the subset of D that contains positive values from vertex issues and let S be the corresponding set of vertices in G. We claim that S forms a vertex cover of G. Towards a contradiction, say that S is not a vertex cover. Then, there exists an edge $e_t = (u, v)$ such that both $d_u, d_v \notin D_V$, thus d_u and d_v are still feasible alternatives, after the deletions we have made. Note that due to the budget constraint, it holds that $|D''| \leq k$. Hence out of the 2(k + 1) issues that the edge voters corresponding to e_t care about, there exists an index ℓ , such that both $e_{t,\ell}$ and $e'_{t,\ell}$ are still available, after the deletion of D. But then, the outcome described in Equation (2) is still feasible, which contradicts the fact that the designated outcome became the unique winner after the deletion of D.

Moving to CDA-1 and CAA-1 we show that we can have hardness results, but only under a non-constant domain size for at least one issue. The proof of Theorem 13 below, shows a connection with some natural problems on graphs, that have been previously linked to election control for other voting rules (Betzler & Uhlmann, 2009).

Theorem 13. CAA-1 and CDA-1 are NP-hard, when $\Delta_i \leq 1$ for every voter *i*, and even when the treewidth of the global dependency graph is at most one, but with non-constant domain size in at least one issue.

Proof. We will firstly prove the hardness of CDA-1. We will perform a reduction from the NP-hard problem MAX OUT-DEGREE DELETION (MOD) (Betzler & Uhlmann, 2009).

Instance: A directed graph G = (V, E), a special vertex $p \in V$ and an integer $k \ge 1$. **Output:** Does there exist $V' \subseteq V$ with $|V'| \le k$ such that p is the only vertex of maximum out-degree in $G[V \setminus V']$?

For $S \subseteq V$, we denote by $deg_S(u)$ the out-degree of vertex u in a graph G = (V, E), when we count only outgoing edges towards the vertices of S. Let P = (G = (V, E), p, k)be an instance of MOD in a directed graph with n vertices and m edges. We create a CDA-1 instance, where we have one issue I_j for every vertex $v_j, j \in [n]$ and an extra issue I_0 , hence $I = \{I_0, I_1, I_2, \ldots, I_n\}$. For $j \in [n]$, the domain of issue I_j is binary in the form $D_j = \{d_j, \overline{d_j}\}$. The domain of I_0 , say D_0 , contains (k+1)(n-1) + 1 alternatives. In particular, it contains an alternative b_p that corresponds to the designated vertex $p \in V$, and for every vertex $v \in V \setminus \{p\}$, there are k+1 alternatives b_v^{ℓ} , for $\ell \in [k+1]$. Essentially, these are identical k+1 'copies' encoding the selection of v in I_0 , and play a significant role in the reverse direction of the reduction. As for the voters, there are two types of voters, edge voters and vertex voters. There is one edge voter for every edge $(i, j) \in E$, with a dependency graph having one edge from I_j to I_0 , and voting as follows:

- For the issue I_0 , she votes conditioned on I_j for $\{d_j : b_i\}$ if i = p or otherwise for $\{d_j : b_i^\ell\}, \forall \ell \in [k+1].$
- For all other issues she is satisfied with any alternative.

For every vertex other than p, we also have a block of L identical voters, where it suffices to take L = m + 1. Each voter in the *j*-th block, with $j \in V \setminus \{p\}$ has a dependency graph with 1 edge, from I_0 to I_j and votes as follows:

- For the issue I_j , she is satisfied with the combinations $\{b_j^{\ell} : d_j\}$ for any ℓ . Also, if the value of I_0 differs from b_j^{ℓ} , for any ℓ , she is satisfied with any value on I_j . Hence, the only restriction is that when the value of I_0 comes from an alternative corresponding to vertex j, the voter can be satisfied with respect to I_j only by d_j .
- For all other issues, she is satisfied with any alternative.

In total, we have m + (n-1)L voters. We also use k as the quota parameter, and we suppose the controller wants to enforce the alternative b_p at issue I_0 . Clearly, for every voter $i, \Delta_i \leq 1$ in her dependency graph, and the global dependency graph is a star centered on I_0 . The maximum domain cardinality is $\mathcal{O}(kn) = \mathcal{O}(n^2)$.

For a better view of the construction we comment on Figure 3, which illustrates the reduction to CDA-1. In particular, it illustrates only a part of the construction that pertains to the vertices of a subgraph G' of the initial graph G given in the instance of MOD (as shown in the upper-left part of the figure). The figure also depicts the voters' ballots (rightmost part of the figure) and the global dependency graph which emerges (lower-left part of the figure). To be more precise, the lower-left part shows the union of the dependency graphs of all voters, where both orientations are present for the edges shown. Hence, the global dependency graph is simply a star centered on I_0 . The connections in the rightmost part of the figure represent acceptable pairs of alternatives by voters. More precisely, a dotted connection between the alternatives d_j and b_j^{ℓ} for some j and ℓ , represents the conditional approval ballot $\{b_j^{\ell} : d_j\}$ of the block of the L identical vertex voters that correspond to v_j of G'. A solid connection between the alternatives d_j and b_i^{ℓ} (resp. $\{d_j : b_p\}$) of an edge voter corresponding to edge (v_i, v_j) (resp. (p, v_j)) of G'.

Suppose there exists a set S of vertices in G of size at most k, say without loss of generality that $S = \{1, \ldots, k\} \subseteq V$, whose deletion leaves p as the only vertex of maximum out-degree. We now choose to delete the corresponding alternatives $\{d_1, \ldots, d_k\}$ from the issues $\{I_1, \ldots, I_k\}$. If we select b_p for the issue I_0 , then the total dissatisfaction score can be brought down to $m - deg_{V \setminus S}(p)$ by choosing d_j for every issue I_j where d_j has not been

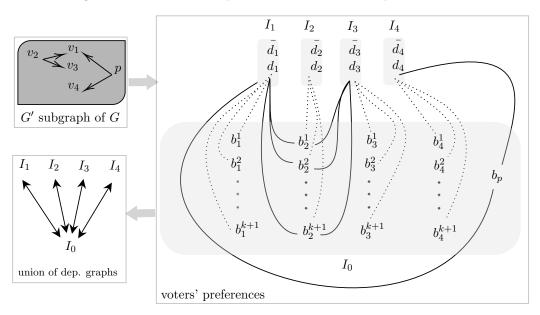


Figure 3: Illustrative example of the reduction in the proof of Theorem 13

deleted. To see this, the only edge voters that are satisfied with respect to I_0 are edges that are outgoing from p and whose other endpoint belongs to $V \setminus S$. Hence all remaining $m - deg_{V \setminus S}(p)$ voters will be dissatisfied with respect to I_0 . Regarding the vertex voters, they will all be satisfied on all issues.

On the other hand, if we select for I_0 some b_j^{ℓ} for any $\ell \in [k + 1]$, we need to consider two cases, depending on j. If $j \in V \setminus S$, then by the same reasoning as before, the best we could achieve is to have a dissatisfaction score equal to $m - deg_{V \setminus S}(j)$. But since p has the maximum out-degree, this would yield a worse solution. Now suppose $j \in S$. Then we know that d_j has been deleted from I_j . Hence, the j-th block of vertex voters will be dissatisfied with respect to I_j , and since L > m, this cannot yield an optimal solution. To conclude, after the deletion of the selected alternatives, b_p has to be selected for I_0 in any optimal solution.

For the reverse direction, suppose that there is a set D'' of at most k alternatives, the deletion of which, forces b_p to be selected for I_0 in every optimal solution. It is without loss of generality to assume that D'' does not contain anything from D_0 . To elaborate on this claim, since there are k + 1 copies of alternatives for every $i \in V \setminus \{p\}$ that have an identical role, there is no change in the optimal outcome by deleting up to k alternatives from I_0 (some alternative will survive for every i). Moreover, we can assume that none of the deleted alternatives equals $\overline{d_j}$ for some issue $I_j \neq I_0$ since if it were, we can swap it with d_j without harming the cost of the optimal solution (one cannot strengthen the support of b_p in I_0 by deleting $\overline{d_j}$ for some j). Also, bear in mind that we are not allowed to delete both d_j and $\overline{d_j}$ from an issue $I_j, j \in [n]$, as there are no other choices left for I_j .

To summarize, the deleted alternatives must come from distinct issues among I_1, \ldots, I_n and they all correspond to some d_j for $j \in [n]$. It is now easy to observe that deleting from V the set S formed by the vertices corresponding to these alternatives in D'', makes p the unique vertex of maximum out-degree in the induced subgraph of G. If not, there is a vertex, say $v \in V \setminus S$, with greater or equal out-degree. In that case, if we select $b_v^{\ell'}$ for I_0 for some arbitrary ℓ' , and d_j for all issues I_j , for which d_j has not been deleted, we will obtain a solution with at most the same dissatisfaction score as the optimal solution that used b_p . Indeed, we will have fewer or equal dissatisfactions from the edge voters with respect to I_0 , and also all the blocks of the vertex voters will be satisfied (the block of voters who care about I_v is satisfied because d_v has not been deleted, since $v \in V \setminus S$). This contradicts the fact that b_p was elected for I_0 in every optimal solution.

For the NP-hardness of CAA-1, the proof is based on a similar reasoning as in the proof of CDA-1, but with appropriate adjustments. First, it is more convenient to perform a reduction from a slightly different problem, which is the MAX-OUTDEGREE ADDITION (MOA) problem defined and proved NP-hard in (Betzler & Uhlmann, 2009).

Instance: A directed graph $G = (V_1 \cup V_2, E)$, where V_1 denotes the set of registered vertices, and V_2 is the set of unregistered vertices, a distinguished vertex $p \in V_1$ and an integer $k \ge 1$.

Output: Does there exist a set $V' \subseteq V_2$ with $|V'| \leq k$ such that p is the only vertex that has maximum outdegree in $G[V_1 \cup V']$?

Starting from an instance of MOA, where $n = |V_1| + |V_2|$, let $I = \{I_0, I_1, I_2, \ldots, I_n\}$. For $j \in V_1$, we have two qualified alternatives, $D_j = \{d_j, \overline{d_j}\}$ and no spoiler ones. For $j \in V_2$, we have one qualified alternative⁷, $D_j = \{\overline{d_j}\}$, and we will have d_j as a spoiler alternative, $D'_j = \{d_j\}$. The domain of I_0 corresponds to all the vertices and equals $D_0 = \{b_1, \ldots, b_n\}$. In contrast to CDV-1, we do not need to have k + 1 "copies" for each b_i , since the spoiler alternatives that will be added are not going to be from D_0 . As for the voters, there is one edge voter for every edge of the graph, regardless of whether its endpoints belong to V_1 or V_2 and one vertex voter for every vertex of the graph. All voters have similar preferences as in the CDA-1 reduction, from which their ballots for each issue I_j with $j \in \{0, 1, \ldots, n\}$ can be immediately obtained by replacing, $\{b_j^\ell\}_{\forall \ell \in [k+1]}$ with b_j . For example, an edge voter arising from an edge (i, j) will vote for the combination $\{d_j : b_i\}$ regarding I_0 . Using similar arguments as in the proof for CDA-1, we conclude that there is a way to add up to k vertices and make p the unique vertex with maximum out-degree if and only if there is a set of at most k alternatives to add in the CAA-1 instance to fulfill controller's will.

Notably, moving to a constant domain size, the considered problems, CDA-1 and CAA-1, seem to behave differently, as the following result indicates.

Proposition 3. CDA-1 can be solved in polynomial time, when $\Delta_i \leq 1$ for every voter i, the treewidth of the global dependency graph is constant and the domain size is also constant for every issue.

Proof. Let q be the quota parameter and let I_j be the issue where the controller wants to enforce a specific alternative. If $q \ge |D_j| - 1$, then we can simply delete precisely all other $|D_j| - 1$ alternatives of I_j so that the controller's will is the only choice left. If $q < |D_j| - 1$, this implies that $q = \mathcal{O}(1)$. Then we can check all possible ways of picking up to q items from the available set of all alternatives of all issues (a polynomial in m). For every such combination, and since the conditions of Theorem 1 hold, we can solve the remaining CMS

^{7.} If one wishes to avoid issues with unary starting domain, we can also add one dummy qualified alternative, so that no issue is trivialized before the addition of any spoiler alternatives.

instance and check if we can have the controller's choice in every optimal solution, by solving CMS with and without the designated alternative. \Box

Hence, a constant domain size makes a difference for CDA-1 when we stick to the assumptions from Section 3.1 on each Δ_i and on the treewidth. For CAA-1, we are not yet aware if the same result holds (the proof arguments certainly do not go through), and we leave this as an interesting open problem. However, we have established intractability, as soon as we move to slightly richer instances with $\Delta_i \leq 2$.

Theorem 14. CAA-1 is NP-hard, when $\Delta_i \leq 2$ for every voter *i*, even when the treewidth of the global dependency graph is at most one and even for a binary domain size in every issue.

Proof. Consider an instance P of VERTEX COVER, asking if there is a cover of size at most kin a graph G = (V, E), with |V| = n, |E| = m. We create an instance P' of CAA-1 with n+1issues $I = \{I_0, I_1, \ldots, I_n\}$. The issue I_0 has two qualified alternatives, $D_0 = \{d_0, \overline{d_0}\}$. Each issue I_j for $j \in [n]$ corresponds to a vertex of G, and has one qualified alternative, denoted by d_j , and one unqualified one denoted by d_j . Formally, $D_j = \{d_j\}$ and $D'_j = \{d_j\}$, for $j \in [n]$. As for the voters, we have a total of 2m-1 voters. The first m voters correspond to the edges of G, and they are satisfied with all the alternatives in the issues $I_i, j \in [n]$. For issue I_0 , each edge voter has a dependence on the two issues corresponding to its endpoints. In particular, for an edge (j, ℓ) , the corresponding edge voter has a dependence of I_0 on both I_j and I_ℓ . He is satisfied with respect to I_0 , only when either d_j or d_ℓ is selected, and d_0 is selected as well. Thus he is satisfied with the combinations $\{(d_i, x) : d_0\}$ for any $x \in \{d_{\ell}, d_{\ell}\}$, and with $\{(x, d_{\ell}) : d_0\}$ for any $x \in \{d_j, d_j\}$. These together encode precisely the constraint $(d_j \vee d_\ell) : d_0$. Any other combination of alternatives of I_j, I_ℓ , and I_0 make this edge voter dissatisfied with respect to I_0 . The remaining m-1 dummy voters are satisfied with all the alternatives of the first n issues and are also satisfied only with d_0 for issue I_0 . To complete the construction, we use k from P as the quota of P', and we assume that the controller wants to enforce d_0 on issue I_0 . It is easy to check that the maximum in-degree for every voter is at most two, and that the global dependency graph is a star centered on I_0 , and hence with treewidth equal to one.

Suppose that P has a vertex cover S of size at most k. We then add in P' the unqualified alternatives for the issues that belong to the vertex cover of G. By selecting those alternatives, and with d_0 for I_0 , and any alternative for the remaining issues, we claim that all the edge voters are satisfied with respect to I_0 (since for every edge, at least one of the added alternatives together with d_0 satisfy the corresponding constraint). Thus, there is only 1 unit of dissatisfaction from every dummy voter on I_0 , with a total score of m - 1. Any solution where d_0 is not the selected choice for I_0 would dissatisfy all the edge voters, and would have a score of at least m, hence cannot be optimal. Thus, we have ensured that in every optimal solution, I_0 is assigned the value of d_0 .

For the reverse direction, suppose that there is a set of at most k unqualified alternatives that, when added, ensure that d_0 is selected in every optimal solution. We know that selecting d_0 causes the dummy voters to be dissatisfied, hence the optimal dissatisfaction score is at least m - 1. If $\overline{d_0}$ was chosen for I_0 , we know that the total dissatisfaction score is m (due to the edge voters), and since this cannot be optimal, we have that the dissatisfaction score in an optimal solution is exactly m-1. But this means that all the remaining m voters, or equivalently all edge voters, have to be satisfied with all issues in the optimal solution, i.e., satisfied with I_0 as well. Thus, the added alternatives need to satisfy every edge voter, which means that if a voter's dependence of I_0 is based on issues I_j and I_ℓ , then either d_j or d_ℓ has been added (or both), and hence the set of added alternatives correspond to a vertex cover of size at most k.

Overall, we end this subsection by concluding that CMS is indeed sufficiently robust against malicious actions, in most of the variants of the control problem considered. The behavior regarding the addition or deletion of alternatives does exhibit differences with that of deleting or adding voters. In some cases, such as in adding alternatives, we get the stronger guarantee of immunity, compared to NP-hardness. On the other hand, when deleting alternatives, we have vulnerability in more settings than when deleting voters. Still, we do have computational hardness, when the domain size is large enough.

5. Conclusions and Further Work

Our work is centered around the CMS rule, a relatively new and highly natural voting rule for expressing preferential dependencies with approval-based conditional ballots in elections over multiple interdependent issues. We focused on computational aspects of CMS elections, from the perspective of the winner determination problem using exact (polynomial and parameterized) and approximate algorithms, as well as from the perspective of strategic attempts to control election outcomes. We conclude that CMS provides a satisfactory tradeoff between expressiveness and efficiency under certain assumptions and at the same time exhibits sufficient robustness against control actions in the considered settings.

There are still several interesting problems for future research. It is conceivable that approximation guarantees can be obtained for instances with higher expressiveness (i.e., higher in-degrees) than those considered in Section 3.2. Additionally, one can also consider other objective functions, such as the Conditional Minimax rule, defined also in (Barrot & Lang, 2016), for which, algorithmic results remain elusive. In principle, one can take any other voting rule defined for approval ballots and explore potential generalizations in the setting with conditional approval ballots, as done for instance in (Brill, Markakis, Papasotiropoulos, & Peters, 2023) with Proportional Approval Voting rule (PAV) and Method of Equal Shares (MES). Furthermore, from a strategic point of view, there is one case of our control questions that has been left open, namely, the complexity of CAA-1, under a binary domain size and with $\Delta_i \leq 1$. Even more interestingly, one can study other strategic moves such as destructive versions of control or bribery in a CMS election, or consider a study on instances that are not worst-case. Along the same spirit, CMS was proven to be non-strategyproof by (Barrot & Lang, 2016), but the complexity of manipulating has not been examined yet. Finally, as highly critical areas of future work, we emphasize the importance of obtaining real or synthetic data on elections over interdependent issues (currently nonexistent in public preference data libraries such as (Mattei & Walsh, 2013)) and of generating experimental results that could complement our theoretical findings.

Acknowledgements

We are grateful to the anonymous IJCAI '21 reviewer who pointed out to us the generalization of Theorem 1 to Theorem 3. We are also thankful to the reviewers of IJCAI '20 and IJCAI '21 for their constructive feedback in earlier versions of this work. This research was supported by the Hellenic Foundation for Research and Innovation. The first author was supported by the "1st Call for HFRI Research Projects to support faculty members and researchers and the procurement of high-cost research equipment" (Project Num. HFRI-FM17-3512) and the second author by the "3rd Call for HFRI PhD Fellowships" (Fellowship Number: 5163).

References

- Agarwal, A., Charikar, M., Makarychev, K., & Makarychev, Y. (2005). $O(\sqrt{\log n})$ approximation algorithms for min uncut, min 2cnf deletion, and directed cut problems. In *Proceedings of the ACM symposium on Theory of computing*, pp. 573–581.
- Airiau, S., Endriss, U., Grandi, U., Porello, D., & Uckelman, J. (2011). Aggregating dependency graphs into voting agendas in multi-issue elections. In *Proceedings of the International Joint Conference on Artificial Intelligence*, pp. 18–23.
- Amanatidis, G., Barrot, N., Lang, J., Markakis, E., & Ries, B. (2015). Multiple referenda and multiwinner elections using Hamming distances: Complexity and manipulability. In Proceedings of the International Conference on Autonomous Agents and Multiagent Systems, pp. 715–723.
- Avidor, A., & Zwick, U. (2005). Approximating MIN 2-SAT and MIN 3-SAT. Theory of Computing Systems, 38(3), 329–345.
- Barrot, N., & Lang, J. (2016). Conditional and sequential approval voting on combinatorial domains. In Proceedings of the International Joint Conference on Artificial Intelligence, pp. 88–94.
- Bartholdi III, J. J., Tovey, C. A., & Trick, M. A. (1992). How hard is it to control an election?. Mathematical and Computer Modelling, 16(8-9), 27–40.
- Bazgan, C., & Paschos, V. T. (2003). Differential approximation for optimal satisfiability and related problems. *European Journal of Operational Research*, 147(2), 397–404.
- Betzler, N., & Uhlmann, J. (2009). Parameterized complexity of candidate control in elections and related digraph problems. *Theoretical Computer Science*, 410(52), 5425– 5442.
- Boutilier, C., Brafman, R. I., Domshlak, C., Hoos, H. H., & Poole, D. (2004). CP-nets: A tool for representing and reasoning withconditional ceteris paribus preference statements. *Journal of Artificial Intelligence Research*, 21, 135–191.
- Brams, S. J., & Fishburn, P. C. (1978). Approval voting. American Political Science Review, 72(3), 831–847.
- Brams, S. J., & Fishburn, P. C. (1982). Deducing preferences and choices in the 1980 presidential election. *Electoral Studies*, 1(3), 333–346.

- Brams, S. J., & Fishburn, P. C. (2010). Going from theory to practice: the mixed success of approval voting. In *Handbook on Approval Voting*, pp. 19–37. Springer Science & Business Media.
- Brams, S. J., & Kilgour, D. M. (2015). Satisfaction approval voting. In Mathematical and Computational Modeling, chap. 11, pp. 273–298. Wiley Online Library.
- Brams, S. J., Kilgour, D. M., & Sanver, M. R. (2007). A minimax procedure for electing committees. Public Choice, 3-4(132), 401–420.
- Brandt, F., Conitzer, V., Endriss, U., Lang, J., & Procaccia, A. D. (2016). Handbook of Computational Social Choice. Cambridge University Press.
- Bredereck, R., Faliszewski, P., Niedermeier, R., Skowron, P., & Talmon, N. (2020). Mixed integer programming with convex/concave constraints: Fixed-parameter tractability and applications to multicovering and voting. *Theoretical Computer Science*, 814, 86–105.
- Brill, M., Markakis, E., Papasotiropoulos, G., & Peters, J. (2023). Proportionality guarantees in elections with interdependent issues. In *Proceedings of the 32nd International Joint Conference on Artificial Intelligence (IJCAI)*, p. 4.
- Chen, J., Faliszewski, P., Niedermeier, R., & Talmon, N. (2017). Elections with few voters: Candidate control can be easy. *Journal of Artificial Intelligence Research*, 60, 937– 1002.
- Colley, R., Grandi, U., & Novaro, A. (2021). Smart voting. In Proceedings of the International Joint Conference on Artificial Intelligence, pp. 1734–1740.
- Cuhadaroglu, T., & Lainé, J. (2012). Pareto efficiency in multiple referendum. Theory and Decision, 72(4), 525–536.
- Cygan, M., Fomin, F. V., Kowalik, L., Lokshtanov, D., Marx, D., Pilipczuk, M., Pilipczuk, M., & Saurabh, S. (2015). *Parameterized Algorithms*. Springer.
- Dalla Pozza, G., Pini, M. S., Rossi, F., & Venable, K. B. (2011). Multi-agent soft constraint aggregation via sequential voting. In *Proceedings of the International Joint Conference* on Artificial Intelligence, pp. 159–191.
- Davies, J., Katsirelos, G., Narodytska, N., & Walsh, T. (2011). Complexity of and algorithms for borda manipulation. In *Proceedings of the AAAI Conference on Artificial Intelligence*, pp. 657–662.
- Demange, M., Grisoni, P., & Paschos, V. T. (1998). Differential approximation algorithms for some combinatorial optimization problems. *Theoretical Computer Science*, 209(1-2), 107–122.
- Endriss, U. (2013). Sincerity and manipulation under approval voting. Theory and Decision, 74, 335–355.
- Faliszewski, P., Hemaspaandra, E., & Hemaspaandra, L. A. (2011). Multimode control attacks on elections. Journal of Artificial Intelligence Research, 40, 305–351.
- Faliszewski, P., & Rothe, J. (2016). Control and bribery in voting.. In Handbook of Computational Social Choice, pp. 146–268. Cambridge University Press.

- Freuder, E. (1990). Complexity of k-tree structured constraint satisfaction problems. In Proceedings of the AAAI Conference on Artificial Intelligence, pp. 4–9.
- Freuder, E. C., & Wallace, R. J. (1992). Partial constraint satisfaction. Artificial Intelligence, 58(1-3), 21–70.
- Garg, N., Vazirani, V. V., & Yannakakis, M. (1996). Approximate max-flow min-(multi) cut theorems and their applications. *SIAM Journal on Computing*, 25(2), 235–251.
- Goldstein, A., Kolman, P., & Zheng, J. (2005). Minimum common string partition problem: Hardness and approximations. *Electronic Journal of Combinatorics*, 12.
- Gottlob, G., & Szeider, S. (2008). Fixed-parameter algorithms for artificial intelligence, constraint satisfaction and database problems. *The Computer Journal*, 51(3), 303– 325.
- Grandi, U., & Endriss, U. (2010). Lifting rationality assumptions in binary aggregation. In Proceedings of the AAAI Conference on Artificial Intelligence, pp. 780–785.
- Grohe, M. (2007). The complexity of homomorphism and constraint satisfaction problems seen from the other side. *Journal of the ACM*, 54(1), 1–24.
- Grohe, M., Schwentick, T., & Segoufin, L. (2001). When is the evaluation of conjunctive queries tractable?. In Proceedings of the Symposium on the Theory of Computing, pp. 657–666.
- Gupta, S., Roy, S., Saurabh, S., & Zehavi, M. (2022). Resolute control: Forbidding candidates from winning an election is hard. *Theoretical Computer Science*, 915, 74–89.
- Hemaspaandra, E., Hemaspaandra, L. A., & Rothe, J. (2007). Anyone but him: The complexity of precluding an alternative. Artificial Intelligence, 171(5-6), 255–285.
- Jain, P., Sornat, K., & Talmon, N. (2020). Participatory budgeting with project interactions. In Proceedings of the International Joint Conference on Artificial Intelligence, pp. 386–392.
- Kilgour, D. M. (2010). Approval balloting for multi-winner elections. In Handbook on Approval Voting, pp. 105–124. Springer Science & Business Media.
- Klein, P. N., Plotkin, S. A., Rao, S., & Tardos, E. (1997). Approximation algorithms for steiner and directed multicuts. *Journal of Algorithms*, 22(2), 241–269.
- Knop, D., Koutecký, M., Masarík, T., & Toufar, T. (2019). Simplified algorithmic metatheorems beyond MSO: treewidth and neighborhood diversity. *Logical Methods in Computer Science*, 15(4).
- Koster, A. M. C. A., van Hoesel, S. P. M., & Kolen, A. W. J. (2002). Solving partial constraint satisfaction problems with tree decomposition. *Networks*, 40(3), 170–180.
- Lackner, M., & Skowron, P. (2023). Multi-Winner Voting with Approval Preferences. Springer Briefs in Intelligent Systems. Springer.
- Laffond, G., & Lainé, J. (2009). Condorcet choice and the Ostrogorski paradox. Social Choice and Welfare, 32(2), 317–333.
- Lang, J., Mengin, J., & Xia, L. (2018). Voting on multi-issue domains with conditionally lexicographic preferences. Artificial Intelligence, 265, 18–44.

- Lang, J., & Xia, L. (2009). Sequential composition of voting rules in multi-issue domains. Mathematical Social Sciences, 57(3), 304–324.
- Lang, J., & Xia, L. (2016). Voting in combinatorial domains. In Handbook of Computational Social Choice, pp. 197–222. Cambridge University Press.
- Laslier, J.-F., & Sanver, M. R. (2010). *Handbook on approval voting*. Springer Science & Business Media.
- Liu, H., Feng, H., Zhu, D., & Luan, J. (2009). Parameterized computational complexity of control problems in voting systems. *Theoretical Computer Science*, 410(27-29), 2746–2753.
- Markakis, E., & Papasotiropoulos, G. (2020). Computational aspects of conditional minisum approval voting in elections with interdependent issues. In *Proceedings of the International Joint Conference on Artificial Intelligence*, pp. 304–310.
- Markakis, E., & Papasotiropoulos, G. (2021). Winner determination and strategic control in conditional approval voting. In *Proceedings of the International Joint Conference* on Artificial Intelligence, pp. 342–348.
- Marx, D. (2010). Can you beat treewidth?. Theory of Computing, 6, 85–112.
- Mattei, N., & Walsh, T. (2013). Preflib: A library for preferences http://www.preflib.org. In International Conference on Algorithmic Decision Theory, pp. 259–270.
- Meir, R., Procaccia, A. D., Rosenschein, J. S., & Zohar, A. (2008). Complexity of strategic behavior in multi-winner elections. *Journal of Artificial Intelligence Research*, 33, 149–178.
- Rey, S., Endriss, U., & de Haan, R. (2023). A general framework for participatory budgeting with additional constraints. Social Choice and Welfare, 1, 1–37.
- Robertson, N., & Seymour, P. D. (1986). Graph minors. II. Algorithmic aspects of treewidth. Journal of Algorithms, 7(3), 309–322.
- Samer, M., & Szeider, S. (2010). Constraint satisfaction with bounded treewidth revisited. Journal of Computer and System Sciences, 76(2), 103–114.
- Vazirani, V. V. (2003). Approximation algorithms. Springer Science & Business Media.
- Weber, R. (1978). Comparison of public choice systems. Cowles Foundation Discussion Papers, 732.
- Williamson, D. P., & Shmoys, D. B. (2011). The Design of Approximation Algorithms. Cambridge University Press.
- Xia, L., & Conitzer, V. (2012). Approximating common voting rules by sequential voting in multi-issue domains. In Proceedings of the International Symposium on Artificial Intelligence and Mathematics.
- Yang, Y. (2019). Complexity of manipulating and controlling approval-based multiwinner voting. In Proceedings of the International Joint Conference on Artificial Intelligence, pp. 637–643.