Prove That Addition Commutes

0) $\forall a : \forall b : (a + Sb) = S(a + b)$ **[axiom]**

1) $\forall b : (d + Sb) = S(d + b)$ **[specification of 0, a replaced by d]**

2) $(d + SSc) = S(d + Sc)$ **[specification of 1, b replaced by Sc]**

3) $\forall b : (Sd + Sb) = S(Sd + b)$ **[specification of 0, a replaced by Sd]**

4) $(Sd + Sc) = S(Sd + c)$ **[specification of 3, b replaced by c]**

5) $S(Sd + c) = (Sd + Sc)$ **[symmetry of 4]**

begin supposition

6) $\forall d : (d + Sc) = (Sd + c)$ **[supposition]**

7) $(d + Sc) = (Sd + c)$ **[specification of 6, d replaced by d]**

8) $S(d + Sc) = S(Sd + c)$ **[successor of 7]**

9) $(d + SSc) = S(Sd + c)$ **[transitivity of 2 and 8]**

10) $(d + SSc) = (Sd + Sc)$ **[transitivty of 9 and 5]**

11) $\forall d : (d + SSc) = (Sd + Sc)$ **[generalization of 10]**

end supposition

12) $\langle \forall d : (d + Sc) = (Sd + c) \rightarrow \forall d : (d + SSc) = (Sd + Sc) \rangle$ **[implication]**

13) $\forall c : \langle \forall d : (d + Sc) = (Sd + c) \rightarrow \forall d : (d + SSc) = (Sd + Sc) \rangle$ **[generalization of 12]**

14) $(d + S0) = S(d + 0)$ **[specification of 1, b replaced by 0]**

15) $\forall a : (a + 0) = a$ **[axiom]**

16) $(d + 0) = d$ **[specification of 15, a replaced by d]**

17) $S(d + 0) = Sd$ **[successor of 16]**

18) $(d + S0) = Sd$ **[transitivity of 14 and 17]**

19) $(Sd + 0) = Sd$ **[specification of 15, a replaced by Sd]**

20) $Sd = (Sd + 0)$ **[symmetry of 19]**

21) $(d + S0) = (Sd + 0)$ **[transitivity of 18 and 20]**

22) $\forall d : (d + S0) = (Sd + 0)$ **[generalization of 21]**

23) $\forall c : \forall d : (d + Sc) = (Sd + c)$ **[induction of c on 22 and 13]**

24) $\forall b : (c + Sb) = S(c + b)$ **[specification of 0, a replaced by c]**

25) $(c + Sd) = S(c + d)$ **[specification of 24, b replaced by d]**

26) $\forall b : (d + Sb) = S(d + b)$ **[specification of 0, a replaced by d]**

27) $(d + Sc) = S(d + c)$ **[specification of 26, b replaced by c]**

28) $S(d + c) = (d + Sc)$ **[symmetry of 27]**

29) $\forall d : (d + Sc) = (Sd + c)$ **[specification of 23, c replaced by c]**

30) $(d + Sc) = (Sd + c)$ **[specification of 29, d replaced by d]**

begin supposition

31) $\forall c : (c + d) = (d + c)$ **[supposition]**

32) $(c + d) = (d + c)$ **[specificationf of 31, c replaced by c]**

33) $S(c + d) = S(d + c)$ **[successor of 32]**

34) $(c + Sd) = S(d + c)$ **[transitivity of 25 and 33]**

35) $(c + Sd) = (d + Sc)$ **[transitivity of 34 and 28]**

36) $(c + Sd) = (Sd + c)$ **[transitivity of 35 and 30]**

37) $\forall c : (c + Sd) = (Sd + c)$ **[generalization of 36]**

end supposition

38) $\langle \forall c : (c + d) = (d + c) \rightarrow \forall c : (c + Sd) = (Sd + c) \rangle$ **[implication]**

39) $\forall d : \langle \forall c : (c + d) = (d + c) \rightarrow \forall c : (c + Sd) = (Sd + c) \rangle$ **[generalization of 38]**

40) $(c + 0) = c$ **[specification of 15, a replaced by c]**

41) $\forall b : (0 + Sb) = S(0 + b)$ **[specification of 0, a replaced by 0]**

42) $(0 + Sb) = S(0 + b)$ **[specification of 41. b replaced by b]**

begin supposition

43) $(0 + b) = b$ **[supposition]**

44) $S(0 + b) = Sb$ **[successor of 43]**

45) $(0 + Sb) = Sb$ **[transitivity of 42 and 44]**

end supposition

46) $\langle (0 + b) = b \rightarrow (0 + Sb) = Sb \rangle$ **[implication]**

47) $\forall b : \langle (0 + b) = b \rightarrow (0 + Sb) = Sb \rangle$ **[generalization of 46]**

48) $(0 + 0) = 0$ **[specification of 15, a replaced by 0]**

49) $\forall b : (0 + b) = b$ **[induction of b on 48 and 47]**

50) $(0 + c) = c$ **[specification of 49, b replaced by c]**

51) $c = (0 + c)$ **[symmetry of 50]**

52) $(c + 0) = (0 + c)$ **[transitivity of 40 and 51]**

53) $\forall c : (c + 0) = (0 + c)$ **[generalization of 52]**

54) $\forall d : \forall c : (c + d) = (d + c)$ **[induction of d on 53 and 39]**